Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 5, Article 91, 2003

ON GRÜSS TYPE INEQUALITIES OF DRAGOMIR AND FEDOTOV

J.E. PEČARIĆ AND B. TEPEŠ

FACULTY OF TEXTILE TECHNOLOGY
UNIVERSITY OF ZAGREB,
PIROTTIJEVA 6, 1000 ZAGREB,
CROATIA.
pecaric@mahazu.hazu.hr
URL: http://mahazu.hazu.hr/DepMPCS/indexJP.html

FACULTY OF PHILOSOPHY,
I. LUČIĆA 3, 1000 ZAGREB, CROATIA.

Received 20 May, 2003; accepted 18 September, 2003
Communicated by S.S. Dragomir

ABSTRACT. Weighted versions of Grüss type inequalities of Dragomir and Fedotov are given. Some related results are also obtained.

Key words and phrases: Grüss type inequalities.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In 1935, G. Grüss proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leqslant \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ satisfying the condition

$$\varphi \leqslant f(x) \leqslant \Phi \text{ and } \gamma \leqslant g(x) \leqslant \Gamma \text{ for all } x \in [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

The following result of Grüss type was proved by S.S. Dragomir and I. Fedotov [1]:

ISSN (electronic): 1443-5756

065-03

Theorem 1.1. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is $L-$Lipschitzian on $[a, b]$, i.e.,*

$$|u(x) - u(y)| \leqslant L|x - y| \quad \text{for all } x \in [a, b], \tag{1.3}$$

f is Riemann integrable on $[a, b]$ and there exist the real numbers m, M so that

$$m \leqslant f(x) \leqslant M \quad \text{for all } x \in [a, b]. \tag{1.4}$$

Then we have the inequality

$$\left| \int_a^b f(x)\, du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)\, dt \right| \leqslant \frac{1}{2} L(M - m)(b - a), \tag{1.5}$$

and the constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

The following result of Grüss' type was proved by S.S. Dragomir and I. Fedotov [2]:

Theorem 1.2. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is $L-$lipschitzian on $[a, b]$, and f is a function of bounded variation on $[a, b]$. Denote by $\bigvee_a^b f$ the total variation of f on $[a, b]$. Then the following inequality holds:*

$$\left| \int_a^b u(x)\, df(x) - \frac{f(b) - f(a)}{b - a} \cdot \int_a^b u(x)\, dx \right| \leqslant \frac{1}{2} L(b - a) \bigvee_a^b f. \tag{1.6}$$

The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

Remark 1.3. For other related results see [3].

Let us also state that the weighted version of (1.1) is well known, that is we have with condition (1.2) the following generalization of (1.1):

$$|D(f, g; w)| \leqslant \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma), \tag{1.7}$$

where

$$D(f, g; w) = A(f, g; w) - A(f; w) A(g; w),$$

and

$$A(f; w) = \frac{\int_a^b w(x) f(x)\, dx}{\int_a^b w(x)\, dx}.$$

So, in this paper we shall show that corresponding weighted versions of (1.5) and (1.6) are also valid. Some related results will be also given.

2. Results

Theorem 2.1. *Let $f, u : [a, b] \to \mathbb{R}$ be such that f is Riemann integrable on $[a, b]$ and u is $L-$Lipschitzian on $[a, b]$, i.e. (1.3) holds true. If $w : [a, b] \to \mathbb{R}$ is a positive weight function, then*

$$|T(f, u; w)| \leqslant L \int_a^b w(x) |f(x) - A(f; w)|\, dx, \tag{2.1}$$

where

$$T(f, u; w) = \int_a^b w(x) f(x)\, du(x) - \frac{1}{\int_a^b w(x)\, dx} \int_a^b w(x)\, du(x) \int_a^b w(x) f(x)\, dx. \tag{2.2}$$

Moreover, if there exist the real numbers m, M such that (1.4) is valid, then

$$|T(f, u; w)| \leqslant \frac{L}{2} (M - m) \int_a^b w(x)\, dx. \tag{2.3}$$

Proof. As in [1], we have

$$|T(f,u;w)| = \left|\int_a^b w(x)\,[f(x) - A(f;w)]\,du(x)\right| \leqslant L\int_a^b w(x)\,|f(x) - A(f;w)|\,dx.$$

That is, (2.1) is valid. Furthermore, from an application of Cauchy's inequality we have:

$$|T(f,u;w)| \leqslant L\left(\int_a^b w(x)\,dx \int_a^b w(x)\,(f(x) - A(f;w))^2\,dx\right)^{\frac{1}{2}}, \tag{2.4}$$

from where we obtain

$$|T(f,u;w)| \leqslant L\cdot(D(f,f;w))^{\frac{1}{2}}\cdot\int_a^b w(x)\,dx. \tag{2.5}$$

From (1.7) for $g \equiv f$ we get:

$$(D(f,f;w))^{\frac{1}{2}} \leqslant \frac{1}{2}(\Phi - \varphi). \tag{2.6}$$

Now, (2.4) and (2.5) give (2.3). □

Now, we shall prove the following result.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be $M-$Lipschitzian on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ be $L-$Lipschitzian on $[a,b]$. If $w : [a,b] \to \mathbb{R}$ is a positive weight function, then*

$$|T(f,u;w)| \leqslant L\cdot M\cdot\frac{\int_a^b\int_a^b w(x)\,w(x)\,|x-y|\,dxdy}{\int_a^b w(y)\,dy}. \tag{2.7}$$

Proof. It follows from (2.1)

$$\begin{aligned}|T(f,u;w)| &\leqslant L\cdot\int_a^b w(x)\left|\frac{\int_a^b w(y)\,(f(x) - f(y))\,dy}{\int_a^b w(y)\,dy}\right|dx \\ &\leqslant L\cdot\int_a^b w(x)\,\frac{\int_a^b w(y)\,|f(x) - f(y)|\,dy}{\int_a^b w(y)\,dy}dx \\ &\leqslant L\cdot M\cdot\frac{\int_a^b\int_a^b w(x)\,w(x)\,|x-y|\,dxdy}{\int_a^b w(y)\,dy}.\end{aligned}$$

□

If in the previous result we set $w(x) \equiv 1$, then we can obtain the following corollary:

Corollary 2.3. *Let f and u be as in Theorem 2.2, then,*

$$\left|\int_a^b f(x)\,du(x) - \frac{u(b) - u(a)}{b-a}\int_a^b f(t)\,dt\right| \leqslant \frac{L\cdot M\cdot(b-a)^2}{3}.$$

Proof. The proof follows by the fact that

$$\begin{aligned}\int_a^b \int_a^b |x-y|\,dxdy &= \int_a^b \left(\int_a^b |x-y|\,dx\right) dy \\ &= \int_a^b \left(\int_a^y (y-x)\,dx + \int_y^b (x-y)\,dx\right) dy \\ &= \frac{1}{2}\int_a^b \left((y-a)^2 + (b-y)^2\right) dy \\ &= \frac{1}{3}(b-a)^3 .\end{aligned}$$

□

Theorem 2.4. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is $L-$Lipschitzian on $[a,b]$, and f is a function of bounded variation on $[a,b]$. If $w : [a,b] \to \mathbb{R}$ is a positive weight function, then the following inequality holds:*

$$|T(u,f;w)| \leqslant ML \bigvee_a^b g \leqslant WML \bigvee_a^b f,$$

where $T(u,f;w)$ is defined by (2.2), $g : [a,b] \to \mathbb{R}$ is the function $g(x) = \int_a^x w(t)\,df(t)$,

$$W = \sup_{x\in[a,b]} w(x), \qquad M = \max\left\{\frac{\int_a^b w(t)(b-t)\,dt}{\int_a^b w(t)\,dt}, \frac{\int_a^b w(t)(t-a)\,dt}{\int_a^b w(t)\,dt}\right\},$$

and $\bigvee_a^b g$ and $\bigvee_a^b f$ denote the total variation of g and f on $[a,b]$, respectively.

Proof. We have

$$\begin{aligned}T(u,f;w) &= \int_a^b w(x)u(x)\,df(x) - \frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)\,df(x)\int_a^b w(x)u(x)\,dx \\ &= \int_a^b w(x)\left(u(x) - \frac{\int_a^b w(t)u(t)\,dt}{\int_a^b w(t)\,dt}\right) df(x) \\ &= \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt}\right) w(x)\,df(x).\end{aligned}$$

Using the fact that u is $L-$Lipschitzian on $[a,b]$, we can state that:

$$\begin{aligned}|T(u,f;w)| &= \left|\int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt}\right) w(x)\,df(x)\right| \\ &= \left|\int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt}\right) d\left(\int_a^x w(t)\,df(t)\right)\right| \\ &\leqslant L\sup_{x\in[a,b]}\left(\frac{\int_a^b w(t)|x-t|\,dt}{\int_a^b w(t)\,dt}\right)\bigvee_a^b\left(\int_a^x w(t)\,df(t)\right) \\ &= ML\bigvee_a^b g.\end{aligned}$$

The constant M has the value

$$M = \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)\,|x-t|\,dt}{\int_a^b w(t)\,dt} \right).$$

If we denote a new function $y(x)$ as:

$$y(x) = \int_a^b w(t)\,|x-t|\,dt = \int_a^x w(t)(x-t)\,dt + \int_x^b w(t)(t-x)\,dt,$$

then the first derivative of this function is:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x\int_a^x w(t)\,tdt - \int_a^x tw(t)\,dt + \int_x^b w(t)\,tdt - x\int_x^b w(t)\,dt \right) \\
&= \int_a^x w(t)\,dt + w(x)\,x - w(x)\,x - w(x)\,x - \int_x^b w(t)\,dt + w(x)\,x \\
&= \int_a^x w(t)\,dt - \int_x^b w(t)\,dt;
\end{aligned}$$

and the second derivative is:

$$\frac{d^2y}{dx^2} = w(x) + w(x) = 2w(x) > 0.$$

Obviously f is a convex function, so we have:

$$\begin{aligned}
M &= \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)\,|x-t|\,dt}{\int_a^b w(t)\,dt} \right) \\
&= \sup_{x\in[a,b]} \left(\frac{y(x)}{\int_a^b w(t)\,dt} \right) \\
&= \max\left\{ \frac{\int_a^b w(t)(b-t)\,dt}{\int_a^b w(t)\,dt}, \frac{\int_a^b w(t)(t-a)\,dt}{\int_a^b w(t)\,dt} \right\}.
\end{aligned}$$

That is:

$$\begin{aligned}
|T(u,f;w)| &= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) w(x)\,df(x) \right| \\
&= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) w(x)\,df(x) \right| \\
&\leqslant \int_a^b \frac{\int_a^b w(t)\,|u(x)-u(t)|\,dt}{\int_a^b w(t)\,dt} w(x)\,|df(x)| \\
&\leqslant \sup_{x\in[a,b]} w(x)\ L \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)\,|x-t|\,dt}{\int_a^b w(t)\,dt} \right) \bigvee_a^b f \\
&= WML \bigvee_a^b f.
\end{aligned}$$

□

REFERENCES

[1] S.S. DRAGOMIR AND I. FEDOTOV, An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4) (1998), 287–292.

[2] S.S. DRAGOMIR AND I. FEDOTOV, A Grüss type inequality for mapping of bounded variation and applications to numerical analysis integral and applications for special means, *RGMIA Res. Rep. Coll.*, **2**(4) (1999). [ONLINE: `http://rgmia.vu.edu.au/v2n4.html`].

[3] S.S. DRAGOMIR, New inequalities of Grüss type for the Stieltjes integral, *RGMIA Res. Rep. Coll.*, **5**(4) (2002), Article 3. [ONLINE: `http://rgmia.vu.edu.au/v5n4.html`].