POINCARÉ TYPE INEQUALITIES FOR VARIABLE EXPONENTS

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Abstract: We consider Poincaré type inequalities of integral form for variable exponents. We give conditions under which these inequalities do not hold as well as conditions under which they hold.

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1. Introduction and preliminaries

One of the classical Poincaré inequalities states

$$\int_G |\varphi(x)|^p \, dx \le C(N, p, |G|) \int_G |\nabla\varphi(x)|^p \, dx, \qquad \forall \varphi \in C_0^1(G),$$

where G is a bounded open set in $\mathbb{R}^N$ ($N \ge 1$) and $p \ge 1$.

In Fu [2], this inequality with p replaced by a bounded variable exponent $p(x)$ is given as a lemma. Namely, let $p(x)$ be a bounded measurable function on G such that $p(x) \ge 1$ for all $x \in G$. We shall say that the Poincaré inequality (PI, for short) holds on G for $p(\cdot)$ if there exists a constant $C > 0$ such that

$$\text{(PI)} \qquad \int_G |\varphi(x)|^{p(x)} \, dx \le C \int_G |\nabla\varphi(x)|^{p(x)} \, dx$$

for all $\varphi \in C_0^1(G)$. Fu's lemma asserts that (PI) always holds. However, as was already remarked in [1, pp. 444-445, Example] in the one dimensional case, this is false. We shall give some types of $p(\cdot)$ for which (PI) does not hold.

We remark here that the following norm-form of the Poincaré inequality holds for variable exponents (cf. [3, Theorem 3.10]):

$$\|\varphi\|_{L^{p(\cdot)}(G)} \le C \| |\nabla\varphi| \|_{L^{p(\cdot)}(G)}$$

for all $\varphi \in C_0^1(G)$ provided that $p(x)$ is continuous on $\overline{G}$, where $\|\cdot\|_{L^{p(\cdot)}(G)}$ denotes the (Luxemburg) norm in the variable exponent Lebesgue space $L^{p(\cdot)}(G)$ (see [3] for definition). Thus, our results show that we must distiguish between norm-form and integral-form when we consider the Poincaré inequalities for variable exponents.

We also consider a slightly weaker form: we shall say that the weak Poincaré inequality (wPI, for short) holds on G for $p(\cdot)$ if there exists a constant $C > 0$ such

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that

$$\text{(wPI)} \qquad \int_G |\varphi(x)|^{p(x)}\,dx \le C\left(1 + \int_G |\nabla\varphi(x)|^{p(x)}\,dx\right)$$

for all $\varphi \in C_0^1(G)$. We shall see that this weak Poincaré inequality does not always hold either.

The main purpose of this paper is to give some sufficient conditions on $p(\cdot)$ under which (PI) or (wPI) holds, and our results show that (PI) holds for a fairly large class of non-constant $p(x)$ and (wPI) holds for $p(x)$ in a larger class.

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2. Invalidity of Poincaré type inequalities

For a measurable function $p(x)$ on G and $E \subset G$, let

$$p_E^+ = \operatorname*{ess\,sup}_{x \in E} p(x) \quad \text{and} \quad p_E^- = \operatorname*{ess\,inf}_{x \in E} p(x).$$

Lemma 2.1. *Let $p(x)$ and $q(x)$ be measurable functions on G such that $0 < p_G^- \le p_G^+ < \infty$ and $0 < q_G^- \le q_G^+ < \infty$.*

1. *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $q_K^- > p^+_{G_2 \setminus \overline{G_1}}$, then there exists a sequence $\{\varphi_n\}$ in $C_0^1(G)$ such that*

$$\int_G |\varphi_n(x)|^{q(x)}\,dx \to \infty \qquad \text{and} \qquad \frac{\int_G |\nabla \varphi_n(x)|^{p(x)}\,dx}{\int_G |\varphi_n(x)|^{q(x)}\,dx} \to 0$$

   as $n \to \infty$.

2. *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $q_K^+ < p^-_{G_2 \setminus \overline{G_1}}$, then there exists a sequence $\{\psi_n\}$ in $C_0^1(G) \setminus \{0\}$ such that*

$$\int_G |\nabla \psi_n(x)|^{p(x)}\,dx \to 0 \qquad \text{and} \qquad \frac{\int_G |\nabla \psi_n(x)|^{p(x)}\,dx}{\int_G |\psi_n(x)|^{q(x)}\,dx} \to 0$$

   as $n \to \infty$.

Proof. Choose $\varphi_1 \in C_0^1(G)$ such that $\varphi_1 = 1$ on $\overline{G_1}$ and Spt $\varphi_1 \subset G_2$.
(1) Suppose $q_K^- > p^+_{G_2 \setminus \overline{G_1}}$. For simplicity, write $q_1 = q_K^-$ and $p_2 = p^+_{G_2 \setminus \overline{G_1}}$. Let $\varphi_n = n\varphi_1$, $n = 1, 2, \ldots$. Then

$$\int_G |\nabla \varphi_n|^{p(x)}\,dx = \int_{G_2 \setminus \overline{G_1}} n^{p(x)} |\nabla \varphi_1|^{p(x)}\,dx \le n^{p_2} \int_G |\nabla \varphi_1|^{p(x)}\,dx$$

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and

$$\int_G |\varphi_n|^{q(x)}\,dx \geq \int_K n^{q(x)}\,dx \geq n^{q_1}|K|.$$

These inequalities show that the sequence $\{\varphi_n\}$ has the required properties.
(2) Suppose $q_K^+ < p^-_{G_2\setminus\overline{G_1}}$. Write $q_2 = q_K^+$ and $p_1 = p^-_{G_2\setminus\overline{G_1}}$. Let $\psi_n = (1/n)\varphi_1,\ n = 1, 2, \ldots$. Then

$$\int_G |\nabla\psi_n|^{p(x)}\,dx = \int_{G_2\setminus\overline{G_1}} n^{-p(x)}|\nabla\varphi_1|^{p(x)}\,dx \leq n^{-p_1}\int_G |\nabla\varphi_1|^{p(x)}\,dx$$

and

$$\int_G |\psi_n|^{q(x)}\,dx \geq \int_K n^{-q(x)}\,dx \geq n^{-q_2}|K|.$$

Thus the sequence $\{\psi_n\}$ has the required properties. $\square$

By taking $p(x) = q(x)$ in this lemma, we readily obtain

Proposition 2.2.

1. *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $p_K^- > p^+_{G_2\setminus\overline{G_1}}$, then (wPI) does not hold for $p(\cdot)$ on G.*
2. *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $p_K^+ < p^-_{G_2\setminus\overline{G_1}}$, then (PI) does not hold for $p(\cdot)$ on G.*

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3. Validity of Poincaré Type Inequalities in One-Dimensional Case

We shall say that $f(t)$ on (t_0, t_1) is of type (L) if there is $\tau \in (t_0, t_1)$ such that $f(t)$ is non-increasing on (t_0, τ) and non-decreasing on (τ, t_1).

Proposition 3.1. *Let $N = 1$ and $G = (a, b)$.*

1. *If $p(t)$ is monotone (i.e., non-decreasing or non-increasing) or of type (L) on G, then*

$$\int_a^b |f(t)|^{p(t)}\,dx \le \frac{|G|}{2} + \max(|G|, |G|^{p^+}) \int_a^b |f'(t)|^{p(t)}\,dt$$

for $f \in C_0^1(G)$, where $|G| = b - a$ and $p^+ = p_G^+$.

2. *If $p(t)$ is monotone on G, then*

$$\int_a^b |f(t)|^{p(t)}\,dx \le C \int_a^b |f'(t)|^{p(t)}\,dt$$

for $f \in C_0^1(G)$, where the constant C depends only on p^+ and $|G|$.

Proof. (I) First, we consider the case $G = (0, 1)$. Let $f \in C_0^1(G)$.

(I-1) Suppose $p(t)$ is non-increasing on $(0, \tau)$, $0 < \tau \le 1$. Then, for $0 < t < \tau$,

$$\begin{aligned}|f(t)|^{p(t)} &\le \left(\int_0^t |f'(s)|\,ds\right)^{p(t)} \le \int_0^t |f'(t)|^{p(t)}\,ds \\ &\le \int_0^t \left(1 + |f'(s)|^{p(s)}\right)\,ds \le t + \int_0^1 |f'(s)|^{p(s)}\,ds.\end{aligned}$$

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Hence

$$\int_0^\tau |f(t)|^{p(t)}\,dt \le \frac{\tau^2}{2} + \tau \int_0^1 |f'(s)|^{p(s)}\,ds.$$

Similarly, if $p(t)$ is non-decreasing on $(\tau,\,1)$, $0 \le \tau < 1$, then

$$\int_\tau^1 |f(t)|^{p(t)}\,dt \le \frac{(1-\tau)^2}{2} + (1-\tau) \int_0^1 |f'(s)|^{p(s)}\,ds.$$

Hence, if $p(t)$ is monotone or of type (L) on G, then

$$\int_0^1 |f(t)|^{p(t)}\,dt \le \frac{1}{2} + \int_0^1 |f'(t)|^{p(t)}\,dt. \tag{3.1}$$

(I-2) The case $\|f'\|_1 := \int_0^1 |f'(t)|\,dt \ge 1$.
In this case,

$$\begin{aligned} 1 &\le \int_0^1 |f'(t)|\,dt = \frac{1}{2}\int_0^1 |2f'(t)|\,dt \\ &\le \frac{1}{2} + \frac{1}{2}\int_0^1 |2f'(t)|^{p(t)}\,dt \le \frac{1}{2} + 2^{p^+-1}\int_0^1 |f'(t)|^{p(t)}dt, \end{aligned}$$

so that

$$\frac{1}{2} \le 2^{p^+-1}\int_0^1 |f'(t)|^{p(t)}dt.$$

Hence, by (3.1), we have

$$\int_0^1 |f(t)|^{p(t)}\,dt \le \left(1 + 2^{p^+-1}\right)\int_0^1 |f'(t)|^{p(t)}dt \tag{3.2}$$

in case $\|f'\|_1 \ge 1$.

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(I-3) The case $p(t)$ is monotone and $\|f'\|_1 < 1$.
We may assume that $p(t)$ is non-decreasing. Set

$$E_1 = \{t \in (0,1);\ |f'(t)| \le 1\}, \quad E_2 = \{t \in (0,1);\ |f'(t)| > 1\},$$

$$g_1(t) = \int_{(0,t)\cap E_1} |f'(s)|\, ds \quad \text{and} \quad g_2(t) = \int_{(0,t)\cap E_2} |f'(s)|\, ds.$$

Then for $0 < t < 1$

$$\begin{aligned} |f(t)|^{p(t)} &\le \left(\int_0^t |f'(s)|\, ds\right)^{p(t)} = \left(g_1(t) + g_2(t)\right)^{p(t)} \\ &\le 2^{p^+-1}\left(g_1(t)^{p(t)} + g_2(t)^{p(t)}\right). \end{aligned}$$

Since $p(s) \le p(t)$ for $0 < s < t$ and $|f(s)| \le 1$ for $s \in E_1$,

$$g_1(t)^{p(t)} \le \int_{(0,t)\cap E_1} |f'(s)|^{p(t)}\, ds \le \int_{(0,t)\cap E_1} |f'(s)|^{p(s)}\, ds \le \int_{E_1} |f'(s)|^{p(s)}\, ds.$$

On the other hand, since $g_2(t) \le \|f'\|_1 < 1$ and $|f'(s)| > 1$ for $s \in E_2$,

$$g_2(t)^{p(t)} \le g_2(t) = \int_{(0,t)\cap E_2} |f'(s)|\, ds \le \int_{E_2} |f'(s)|^{p(s)}\, ds.$$

Hence

$$|f(t)|^{p(t)} \le 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)}\, ds$$

for all $0 < t < 1$, and hence

$$\int_0^1 |f(t)|^{p(t)}\, dt \le 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)}\, ds$$

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in case $\|f'\|_1 < 1$.

(I-4) Combining (I-2) and (I-3), we have (3.2) for all $f \in C_0^1(G)$ if $p(t)$ is monotone.

(II) The general case: Let $G = (a,b)$ and $f \in C_0^1(G)$. Let

$$g(t) = f(a + t(b-a)) \quad \text{and} \quad q(t) = p(a + t(b-a))$$

for $0 < t < 1$. Then

$$\int_a^b |f(s)|^{p(s)}\, ds = (b-a) \int_0^1 |g(t)|^{q(t)}\, dt$$

and

$$\begin{aligned} \int_0^1 |g'(t)|\, dt &= \frac{1}{b-a} \int_a^b |(b-a) f'(s)|^{p(s)}\, ds \\ &\le \max(1, (b-a)^{p^+ - 1}) \int_a^b |f'(s)|^{p(s)}\, ds. \end{aligned}$$

Hence, applying (3.1) and (3.2) to $g(t)$ and $q(t)$, we obtain the required inequalities of the proposition. (In fact, we can take $C = (1 + 2^{p^+ - 1}) \max(|G|, |G|^{p^+})$.) $\square$

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4. Validity of Poincaré Type Inequalities in Higher-Dimensional Case

Theorem 4.1. *Let $N \geq 2$ and $G \subset G' \times (a,\, b)$ with a bounded open set $G' \subset \mathbb{R}^{N-1}$ and set $G_{x'} = \{t \in (a,\, b) : (x', t) \in G\}$ for $x' \in G'$.*

1. *If $t \mapsto p(x', t)$ is monotone or of type (L) on each component of $G_{x'}$ for a.e. $x' \in G'$ (with respect to the $(N-1)$-dimensional Lebesgue measure), then (wPI) holds for $p(\cdot)$ on G.*
2. *If $t \mapsto p(x', t)$ is monotone on each component of $G_{x'}$ for a.e. $x' \in G'$ (with respect to the $(N-1)$-dimensional Lebesgue measure), then (PI) holds for $p(\cdot)$ on G.*

Proof. Fix $x' \in G'$ for a moment and let I_j be the components of $G_{x'}$. If $\varphi \in C_0^1(G)$, then $t \mapsto \varphi(x', t)$ belongs to $C_0^1(I_j)$ for each j. Thus, by Proposition 3.1, if $t \mapsto p(x', t)$ is monotone or of type (L) on each I_j, then

$$\int_{I_j} |\varphi(x', t)|^{p(x',t)}\, dt \leq |I_j| + \max(1, |I_j|^{p^+}) \int_{I_j} |\nabla\varphi(x', t)|^{p(x',t)}\, dt,$$

so that

$$\int_{G_{x'}} |\varphi(x', t)|^{p(x',t)}\, dt \leq |G_{x'}| + \max(1, (b-a)^{p^+}) \int_{G_{x'}} |\nabla\varphi(x', t)|^{p(x',t)}\, dt;$$

and if $t \mapsto p(x', t)$ is monotone on each I_j then

$$\int_{I_j} |\varphi(x', t)|^{p(x',t)}\, dt \leq C(p^+, I_j) \int_{I_j} |\nabla\varphi(x', t)|^{p(x',t)}\, dt,$$

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so that

$$\int_{G_{x'}} |\varphi(x',t)|^{p(x',t)}\, dt \leq C(p^+, b-a) \int_{G_{x'}} |\nabla \varphi(x',t)|^{p(x',t)}\, dt.$$

Hence, integrating over G' with respect to x', we obtain the assertion of the theorem. □

The following proposition is easily seen by a change of variables:

Proposition 4.2. *(PI) and (wPI) are diffeomorphically invariant. More precisely, let G_1 and G_2 be bounded open sets and $\Phi(x) = (\phi_1(x), \dots, \phi_N(x))$ be a (C^1-)diffeomorphism of G_1 onto G_2. Suppose $|\nabla \phi_j|$, $j = 1, \dots, N$ and $|\nabla \psi_j|$, $j = 1, \dots, N$ are all bounded, where $\Phi^{-1}(y) = (\psi_1(y), \dots, \psi_N(y))$, and suppose $0 < \alpha \leq J_\Phi(x) \leq \beta$ for all $x \in G_1$. Let $p_1(x) = p_2(\Phi(x))$ for $x \in G_1$. Then, (PI) (resp. (wPI)) holds for $p_1(\cdot)$ on G_1 if and only if it holds for $p_2(\cdot)$ on G_2.*

Combining Theorem 4.1 with this Proposition, we can find a fairly large class of $p(x)$ for which (PI) (as well as (wPI)) holds.

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