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ON THE EXTENDED HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. This paper gives two distinct generalizations of the extended Hilbert's integral inequality with the same best constant factor involving the β function. As applications, we consider some equivalent inequalities.

Key words and phrases: Hilbert's inequality, weight function, β function, Hölder's inequality.

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1. INTRODUCTION

If $f, g \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then the famous Hilbert's integral inequality is given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible (see [2]). Inequality (1.1) had been generalized by Hardy-Riesz [1] as:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. When $p = q = 2$, inequality (1.2) reduces to (1.1). We call (1.2) Hardy-Hilbert's integral inequality, which is important in analysis and its applications (see [4]).

In recent years, by introducing a parameter λ and the β function, Yang [7, 8] gave an extension of (1.2) as:

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If $\lambda > 2 - \min\{p, q\}$, $0 < \int_0^\infty x^{1-\lambda} f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{1-\lambda} g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < k_\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{p+\lambda-2}{p}\right)$ is the best possible ($B(u,v)$ is the β function). Its equivalent inequality is (see [9, (2.12)]):

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right]^p dy < [k_\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x)dx, \tag{1.4}$$

where the constant factor $[k_\lambda(p)]^p = \left[B\left(\frac{p+\lambda-2}{p}, \frac{p+\lambda-2}{p}\right)\right]^p$ is the best possible.

When $\lambda = 1$, inequality (1.3) reduces to (1.2), and (1.4) reduces to the equivalent form of (1.2) as:

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty f^p(x)dx. \tag{1.5}$$

For $p = q = 2$, by (1.3), we have $\lambda > 0$, and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx \right\}^{\frac{1}{2}}. \tag{1.6}$$

We define (1.6) as the extended Hilbert's integral inequality. Recently, Yang et al. [10] provided an extensive account of the above results and Yang [6] gave a reverse of (1.4) with the same best constant factor. The main objective of this paper is to build two distinct generalizations of (1.6), with the same best constant factor but different from (1.3). As applications, we consider some equivalent inequalities.

For this, we need some lemmas.

2. SOME LEMMAS

We have the formula of the β function as (see [5]):

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt = B(v,u) \qquad (u,v>0). \tag{2.1}$$

Lemma 2.1 (see [3]). *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega(\sigma) > 0$, $f, g \geq 0$, $f \in L^p_\omega(E)$ and $g \in L^q_\omega(E)$, then the weighted Hölder's inequality is as follows:*

$$\int_E \omega(\sigma) f(\sigma) g(\sigma) d\sigma \leq \left\{ \int_E \omega(\sigma) f^p(\sigma) d\sigma \right\}^{\frac{1}{p}} \left\{ \int_E \omega(\sigma) g^q(\sigma) d\sigma \right\}^{\frac{1}{q}}, \tag{2.2}$$

where the equality holds if and only if there exists non-negative real numbers A and B, such that they are not all zero and $Af^p(\sigma) = Bg^q(\sigma)$, a.e. in E.

Lemma 2.2. *If $r > 1$, and $\lambda > 0$, define the weight function $\omega_\lambda(r,x)$ as*

$$\omega_\lambda(r,x) := x^{\lambda(1-\frac{1}{r})} \int_0^\infty \frac{1}{(x+y)^\lambda} y^{(\lambda-r)/r} dy. \tag{2.3}$$

Then we have

$$\omega_\lambda(r,x) = B\left(\frac{\lambda}{r}, \lambda\left(1 - \frac{1}{r}\right)\right). \tag{2.4}$$

Proof. Setting $y = xu$ in the integral of (2.3), we find

$$\omega_\lambda(r,x) = x^{\lambda\left(1-\frac{1}{r}\right)} \int_0^\infty \frac{(xu)^{(\lambda-r)/r}}{x^\lambda(1+u)^\lambda} x du = \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda}{r}-1} du.$$

By (2.1), we have (2.4) and the lemma is proved. □

Note. It is obvious that for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda > 0$, one has

$$\omega_\lambda(p,x) = B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) = \omega_\lambda(q,x). \tag{2.5}$$

Lemma 2.3. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $0 < \varepsilon < \lambda$, *one has*

$$I_1 := \int_1^\infty y^{\frac{\lambda-q-\varepsilon}{q}} \int_1^\infty \frac{1}{(x+y)^\lambda} x^{\frac{\lambda-p-\varepsilon}{p}} dxdy > \frac{1}{\varepsilon} B\left(\frac{\lambda-\varepsilon}{p}, \frac{\lambda}{q} + \frac{\varepsilon}{p}\right) - \left(\frac{p}{\lambda-\varepsilon}\right)^2. \tag{2.6}$$

Proof. Setting $x = yu$ in I_1, in view of (2.1), one has

$$\begin{aligned} I_1 &= \int_1^\infty y^{-1-\varepsilon} \left[\int_{1/y}^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-p-\varepsilon}{p}} du\right] dy \\ &= \int_1^\infty y^{-1-\varepsilon} \left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\varepsilon}{p}-1} du\right] dy \\ &\quad - \int_1^\infty y^{-1-\varepsilon} \left[\int_0^{\frac{1}{y}} \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\varepsilon}{p}-1} du\right] dy \\ &> \frac{1}{\varepsilon} B\left(\frac{\lambda-\varepsilon}{p}, \frac{\lambda}{q} + \frac{\varepsilon}{p}\right) - \int_1^\infty y^{-1} \int_0^{\frac{1}{y}} u^{\frac{\lambda-\varepsilon}{p}-1}\, dudy. \end{aligned}$$

By calculating the above integral, one has (2.6). The lemma is proved. □

Lemma 2.4. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $0 < \varepsilon < \lambda(p-1)$, *one has*

$$I_2 := \int_1^\infty y^{\lambda-\frac{\lambda+\varepsilon}{q}-1} \int_1^\infty \frac{1}{(x+y)^\lambda} x^{\lambda-\frac{\lambda+\varepsilon}{p}-1} dxdy > \frac{1}{\varepsilon} B\left(\frac{\lambda}{q} - \frac{\varepsilon}{p}, \frac{\lambda}{p} + \frac{\varepsilon}{p}\right) - \left(\frac{\lambda}{q} - \frac{\varepsilon}{p}\right)^{-2}. \tag{2.7}$$

Proof. Setting $x = yu$ in I_2, in view of (2.1), one has

$$\begin{aligned} I_2 &= \int_1^\infty y^{-1-\varepsilon} \left[\int_{1/y}^\infty \frac{1}{(1+u)^\lambda} u^{\lambda-\frac{\lambda+\varepsilon}{p}-1} du\right] dy \\ &= \int_1^\infty y^{-1-\varepsilon} \left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda-\frac{\lambda+\varepsilon}{p}-1} du\right] dy \\ &\quad - \int_1^\infty y^{-1-\varepsilon} \left[\int_0^{\frac{1}{y}} \frac{1}{(1+u)^\lambda} u^{\lambda-\frac{\lambda+\varepsilon}{p}-1} du\right] dy \\ &> \frac{1}{\varepsilon} B\left(\frac{\lambda}{q} - \frac{\varepsilon}{p}, \frac{\lambda}{p} + \frac{\varepsilon}{p}\right) - \int_1^\infty y^{-1} \int_0^{\frac{1}{y}} u^{\lambda-\frac{\lambda+\varepsilon}{p}-1}\, dudy. \end{aligned}$$

By calculating the above integral, one has (2.7). The lemma is proved. □

3. MAIN RESULTS AND APPLICATIONS

Theorem 3.1. *If* $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, *such that* $0 < \int_0^\infty x^{p-1-\lambda} f^p(x)dx < \infty$ *and* $0 < \int_0^\infty x^{q-1-\lambda} g^q(x)dx < \infty$*, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_0^\infty x^{p-1-\lambda} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q-1-\lambda} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{3.1}$$

$$\int_0^\infty y^{\lambda(p-1)-1} \left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx\right]^p dy < \left[B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_0^\infty x^{p-1-\lambda} f^p(x)dx, \tag{3.2}$$

where the constant factors $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ *and* $\left[B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p$ *are all the best possible. Inequality (3.2) is equivalent to (3.1). In particular, for* $\lambda = 1$*, one has the following two equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty x^{p-2} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q-2} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{3.3}$$

$$\int_0^\infty y^{p-2} \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty x^{p-2} f^p(x)dx. \tag{3.4}$$

Proof. By (2.2), one has

$$\begin{aligned}
J_1 &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \\
&= \int_0^\infty \int_0^\infty \frac{1}{(x+y)^\lambda} \left[\left(\frac{x^{p-\lambda}}{y^{q-\lambda}}\right)^{\frac{1}{pq}} f(x)\right] \left[\left(\frac{y^{q-\lambda}}{x^{p-\lambda}}\right)^{\frac{1}{pq}} g(y)\right] dxdy \\
&\leq \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \left(\frac{x^{p-\lambda}}{y^{q-\lambda}}\right)^{\frac{1}{q}} dy\right] f^p(x)dx\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \left(\frac{y^{q-\lambda}}{x^{p-\lambda}}\right)^{\frac{1}{p}} dx\right] g^q(y)dy\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3.5}$$

If (3.5) takes the form of an equality, then by Lemma 2.1, there exist real numbers A and B, such that they are not all zero, and

$$A \frac{1}{(x+y)^\lambda} \left(\frac{x^{p-\lambda}}{y^{q-\lambda}}\right)^{\frac{1}{q}} f^p(x) = B \frac{1}{(x+y)^\lambda} \left(\frac{y^{q-\lambda}}{x^{p-\lambda}}\right)^{\frac{1}{p}} g^q(y), \quad \text{a.e. in } (0,\infty) \times (0,\infty).$$

Hence we find

$$Ax^{p-\lambda} f^p(x) = By^{q-\lambda} g^q(y), \text{ a.e. in } (0,\infty) \times (0,\infty).$$

It follows that there exists a constant C, such that

$$Ax^{p-\lambda}f^p(x) = C, \quad \text{a.e. in } (0,\infty);$$
$$By^{q-\lambda}g^q(y) = C, \quad \text{a.e. in } (0,\infty).$$

Without loss of generality, suppose that $A \neq 0$. One has

$$x^{p-\lambda-1}f^p(x) = \frac{C}{Ax}, \quad \text{a.e. in } (0,\infty),$$

which contradicts the fact that $0 < \int_0^\infty x^{p-1-\lambda}f^p(x)dx < \infty$. Hence, (3.5) takes the form of strict inequality, and by (2.3), we may rewrite (3.5) as

$$J_1 < \left\{\int_0^\infty \omega_\lambda(q,x)x^{p-1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \omega_\lambda(p,y)y^{q-1-\lambda}g^q(y)dy\right\}^{\frac{1}{q}}. \tag{3.6}$$

Hence by (2.5), one has (3.1).

For $0 < \varepsilon < \lambda$, setting $\widetilde{f}(x)$ and $\widetilde{g}(y)$ as:

$$\widetilde{f}(x) = \widetilde{g}(y) = 0, \qquad x, y \in (0,1);$$
$$\widetilde{f}(x) = x^{\frac{\lambda-p-\varepsilon}{p}}, \widetilde{g}(y) = y^{\frac{\lambda-q-\varepsilon}{q}}, \qquad x, y \in [1,\infty),$$

then we find

$$\left\{\int_0^\infty x^{p-1-\lambda}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q-1-\lambda}\widetilde{g}^q(x)dx\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{3.7}$$

If there exists $\lambda > 0$, such that the constant factor in (3.1) is not the best possible, then there exists a positive number K (with $K < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$), such that (3.1) is still valid if one replaces $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ by K. In particular, one has

$$\begin{aligned} \varepsilon I_1 &= \varepsilon \int_0^\infty \int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(y)}{(x+y)^\lambda} dxdy \\ &< \varepsilon K \left\{\int_0^\infty x^{p-1-\lambda}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q-1-\lambda}\widetilde{g}^q(x)dx\right\}^{\frac{1}{q}}. \end{aligned}$$

Hence by (2.6) and (3.7), one has

$$B\left(\frac{\lambda-\varepsilon}{p}, \frac{\lambda}{q} + \frac{\varepsilon}{p}\right) - \varepsilon\left(\frac{p}{\lambda-\varepsilon}\right)^2 < K,$$

and then $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \leq K \ \ (\varepsilon \to 0^+)$. This contradicts the fact that $K < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$. It follows that the constant factor in (3.1) is the best possible.

Since $0 < \int_0^\infty x^{p-1-\lambda}f^p(x)dx < \infty$, there exists $T_0 > 0$, such that for any $T > T_0$, one has $0 < \int_0^T x^{p-1-\lambda}f^p(x)dx < \infty$. We set

$$g(y,T) := y^{\lambda(p-1)-1}\left[\int_0^T \frac{f(x)}{(x+y)^\lambda}dx\right]^{p-1},$$

and use (3.1) to obtain

$$0 < \int_0^T y^{q-1-\lambda} g^q(y,T)dy$$
$$= \int_0^T y^{\lambda(p-1)-1} \left[\int_0^T \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy$$
$$= \int_0^T \int_0^T \frac{f(x)g(y,T)}{(x+y)^\lambda}dxdy$$
$$< B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\int_0^T x^{p-1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^T y^{q-1-\lambda}g^q(y,T)dy\right\}^{\frac{1}{q}}. \tag{3.8}$$

Hence we find

$$0 < \left[\int_0^T y^{q-1-\lambda} g^q(y,T)dy\right]^{1-\frac{1}{q}}$$
$$= \left\{\int_0^T y^{\lambda(p-1)-1} \left[\int_0^T \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\int_0^T x^{p-1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{3.9}$$

It follows that $0 < \int_0^\infty y^{q-1-\lambda} g^q(y,\infty)dy < \infty$. Hence (3.8) and (3.9) are strict inequalities as $T \to \infty$. Thus inequality (3.2) holds.

On the other hand, if (3.2) is valid, by Hölder's inequality (2.2), one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$
$$= \int_0^\infty \left[y^{\frac{\lambda+1-q}{q}}\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]\left[y^{-\frac{\lambda+1-q}{q}}g(y)\right]dy$$
$$\le \left\{\int_0^\infty y^{\lambda(p-1)-1}\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q-1-\lambda}g^q(y)dy\right\}^{\frac{1}{q}}. \tag{3.10}$$

Hence by (3.2), one has (3.1). It follows that (3.2) is equivalent to (3.1).

If the constant factor in (3.2) is not the best possible, one can get a contradiction that the constant factor in (3.1) is not the best possible by using (3.10). The theorem is thus proved. □

Theorem 3.2. *If $f, g \ge 0$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 0$, such that $0 < \int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{(q-1)(1-\lambda)}g^q(x)dx < \infty$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{(q-1)(1-\lambda)}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{3.11}$$

$$\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy < \left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p \int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx, \tag{3.12}$$

where the constant factors $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ and $\left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p$ are the best possible. Inequality (3.12) is equivalent to (3.11). In particular, for $\lambda = p > 1$, one has the following two equivalent *inequalities:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^p}dxdy < \frac{1}{p-1}\left\{\int_0^\infty \frac{f^p(x)}{x^{(p-1)^2}}dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \frac{g^q(x)}{x}dx\right\}^{\frac{1}{q}} \tag{3.13}$$

and

$$\int_0^\infty y^{p-1}\left[\int_0^\infty \frac{f(x)}{(x+y)^p}dx\right]^p dy < \left(\frac{1}{p-1}\right)^p \int_0^\infty \frac{f^p(x)}{x^{(p-1)^2}}dx. \tag{3.14}$$

Proof. By (2.2), one has

$$\begin{aligned} J_1 &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ &= \int_0^\infty \int_0^\infty \frac{1}{(x+y)^\lambda}\left[\left(\frac{x^{(q-\lambda)/q^2}}{y^{(p-\lambda)/p^2}}\right)f(x)\right]\left[\left(\frac{y^{(p-\lambda)/p^2}}{x^{(q-\lambda)/q^2}}\right)g(y)\right]dxdy \\ &\le \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda}\left(\frac{x^{(q-\lambda)p/q^2}}{y^{(p-\lambda)/p}}\right)dy\right]f^p(x)dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda}\left(\frac{y^{(p-\lambda)q/p^2}}{x^{(q-\lambda)/q}}\right)dx\right]g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.15}$$

Following the same manner as (3.6), one has

$$J_1 < \left\{\int_0^\infty \omega_\lambda(p,x)x^{(p-1)(1-\lambda)}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega_\lambda(q,x)x^{(q-1)(1-\lambda)}g^q(x)dx\right\}^{\frac{1}{q}}. \tag{3.16}$$

Hence by (2.5), one has (3.11).

For $0 < \varepsilon < \lambda(p-1)$, setting $\widetilde{f}(x)$ and $\widetilde{g}(y)$ as:

$$\begin{aligned} &\widetilde{f}(x) = \widetilde{g}(y) = 0, \quad x, y \in (0,1); \\ &\widetilde{f}(x) = x^{\lambda-1-\frac{\lambda+\varepsilon}{p}}, \widetilde{g}(y) = y^{\lambda-1-\frac{\lambda+\varepsilon}{q}}, \quad x, y \in [1,\infty), \end{aligned}$$

by Lemma 2.4 and the same way of Theorem 3.1, we can show that the constant factor in (3.11) is the best possible.

In a similar fashion to Theorem 3.1, we can show that (3.12) is valid, which is equivalent to (3.11). By the equivalence of (3.11) and (3.12), we may conclude that the constant factor in (3.12) is the best possible. The theorem is proved. □

Remark 3.3. (i) For $p = q = 2$, both inequalities (3.1) and (3.11) reduce to (1.6). Inequalities (3.1) and (3.11) are distinct generalizations of (1.6) with the same best constant factor $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$, but different from (1.3).

(ii) Since inequalities (3.3) and (1.2) are different, we may conclude that inequality (3.1) is not a generalization of (1.3).

(iii) Since all the given inequalities are with the best constant factors, we have obtained some new results.

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