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MOMENTS INEQUALITIES OF A RANDOM VARIABLE DEFINED OVER A FINITE INTERVAL

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Abstract

Some estimations and inequalities are given for the higher order central moments of a random variable taking values on a finite interval. An application is considered for estimating the moments of a truncated exponential distribution.

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1. Introduction

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine the moments, if they exist. There are, however, applications wherein the exact forms of probability distributions are not known or are mathematically intractable so that the moments can not be calculated. As an example, an application in insurance in connection with the insurer's payout on a given contract or group of contracts follows a mixture or compound probability distribution that may not be known explicitly. It is this problem that motivates to find alternative estimations for the moments of a probability distribution. Based on the mathematical inequalities, we develop some estimations of the moments of a random variable taking its values on a finite interval.

Set X to denote a random variable whose probability function is $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ and its associated distribution function $F : [a, b] \to [0, 1]$.

Denote by M_r the r^{th} central moment of the random variable X defined as

$$M_r = \int_a^b (t - \mu)^r dF, \; r = 0, 1, 2, \dots, \tag{1.1}$$

where μ is the mean of the random variable X. It may be noted that $M_0 = 1$, $M_1 = 0$ and $M_2 = \sigma^2$, the variance of the random variable X.

When reference is made to the r^{th} moment of a particular distribution, we assume that the appropriate integral (1.1) converges for that distribution.

2. Results Involving Higher Moments

We first prove the following theorem for the higher central moments of the random variable X.

Theorem 2.1. *For the random variable X with distribution function $F : [a,b] \to [0,1]$,*

$$\int_a^b (b-t)(t-a)^m dF = \sum_{k=0}^{m} \binom{m}{k} (\mu - a)^k \left[(b-\mu) M_{m-k} - M_{m-k+1}\right], \quad m = 1, 2, 3, \dots \; . \tag{2.1}$$

Proof. Expressing the left hand side of (2.1) as

$$\int_a^b (b-t)(t-a)^m dF = \int_a^b [(b-\mu) - (t-\mu)][(t-\mu) + (\mu - a)]^m dF,$$

and using the binomial expansion

$$[(t-\mu) + (\mu - a)]^m = \sum_{k=0}^{m} \binom{m}{k} (\mu - a)^k (t-\mu)^{m-k},$$

we get

$$\int_a^b (b-t)(t-a)^m dF = \int_a^b [(b-\mu) - (t-\mu)] \left[\sum_{k=0}^{m} \binom{m}{k} (\mu - a)^k (t-\mu)^{m-k}\right] dF$$

$$= \sum_{k=0}^{m} \binom{m}{k} (b-\mu)(\mu-a)^k \cdot \int_a^b (t-\mu)^{m-k} dF$$
$$- \sum_{k=0}^{m} \binom{m}{k} (\mu-a)^k \cdot \int_a^b (t-\mu)^{m-k+1} dF,$$

and hence the theorem. □

In practice numerical moments of order higher than the fourth are rarely considered, therefore, we now derive the results for the first four central moments of the random variable X based on Theorem 2.1.

Corollary 2.2. *For $m = 1, k = 0, 1$ in (2.1), we have*

$$\int_a^b (b-t)(t-a) dF = (b-\mu)(\mu-a) - M_2. \tag{2.2}$$

This is a result in Theorem 1 by Barnett and Dragomir [1].

Corollary 2.3. *For $m = 2, k = 0, 1, 2$ in (2.1),*

$$\int_a^b (b-t)(t-a)^2 dF = (b-\mu)(\mu-a)^2 + [(b-\mu) - 2(\mu-a)] M_2 - M_3. \tag{2.3}$$

Corollary 2.4. *For $m = 3, k = 0, 1, 2, 3$, we have from (2.1)*

$$\begin{aligned} \int_a^b (b-t)(t-a)^3 dF &= (b-\mu)(\mu-a)^3 + 3(\mu-a)[(b-\mu)-(\mu-a)] M_2 \\ &\quad + [(b-\mu) - 3(\mu-a)] M_3 - M_4. \end{aligned} \tag{2.4}$$

3. Some Estimations for the Central Moments

We apply Hölder's inequality [4] and results of Barnett and Dragomir [1] to derive the bounds for the central moments of the random variable X.

Theorem 3.1. *For the random variable X with distribution function $F : [a,b] \to [0,1]$, we have*

$$\int_a^b (b-t)^r (t-a)^s dF \leq \begin{cases} (b-a)^{r+s+1} \cdot \dfrac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \cdot \|f\|_\infty, \\ \\ (b-a)^{2+\frac{1}{q}} [B(rq+1, sq+1)] \cdot \|f\|_p, \end{cases} \tag{3.1}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1, r, s \geq 0$.

Proof. Let $t = a(1-u) + bu$. Then

$$\int_a^b (b-t)^r (t-a)^s dt = (b-a)^{r+s+1} \cdot \int_0^1 (1-u)^r u^s du.$$

Since $\int_0^1 u^s (1-u)^r du = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}$,

$$\int_a^b (b-t)^r (t-a)^s dt = (b-a)^{r+s+1} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}.$$

Using the property of definite integral,

$$\int_a^b (b-t)^s (t-a)^r dF \geq 0, \quad \text{for } r, s \geq 0, \tag{3.2}$$

we get,

$$\int_a^b (b-t)^s (t-a)^r dF$$
$$\leq ||f||_\infty \int_a^b (b-t)^s (t-a)^r dt,$$
$$= (b-a)^{r+s+1} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \cdot ||f||_\infty \text{ for } r, s \geq 0,$$

the first inequality in (3.1).

Now applying the Hölder's integral inequality,

$$\int_a^b (b-t)^s (t-a)^r dF$$
$$\leq \left[\int_a^b f^p(t)dt\right]^{\frac{1}{p}} \left[\int_a^b (b-t)^{sq}(t-a)^{rq}dt\right]^{\frac{1}{q}}$$
$$= (b-a)^{2+\frac{1}{q}} [B(rq+1, sq+1)] \cdot ||f||_p,$$

the second inequality in (3.1). □

Theorem 3.2. *For the random variable X with distribution function $F : [a,b] \to [0,1]$,*

$$m(b-a)^{r+s+1} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \tag{3.3}$$
$$\leq \int_a^b (b-t)^s (t-a)^r dF$$

$$\leq M(b-a)^{r+s+1} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}, r, s \geq 0.$$

Proof. Noting that if $m \leq f \leq M$, a.e. on $[a, b]$, then

$$m(b-t)^s(t-a)^r \leq (b-t)^s(t-a)^r f \leq M(b-t)^s(t-a)^r,$$

a.e. on $[a, b]$ and by integrating over $[a, b]$, we prove the theorem. □

3.1. Bounds for the Second Central Moment M_2 (Variance)

It is seen from (2.2) and (3.2) that the upper bound for M_2, variance of the random variable X, is

$$M_2 \leq (b-\mu)(\mu-a). \tag{3.4}$$

Considering $x = (b-\mu)$ and $y = (\mu - a)$ in the elementary result

$$xy \leq \frac{(x+y)^2}{4}, \quad x, y \in \mathbb{R},$$

we have

$$M_2 \leq \frac{(b-a)^2}{4}, \tag{3.5}$$

and thus,

$$0 \leq M_2 \leq (b-\mu)(\mu-a) \leq \frac{(b-a)^2}{4}. \tag{3.6}$$

From (2.2) and (3.1), we get

$$(b-\mu)(\mu-a)-M_2 \le \frac{(b-a)^3}{6}||f||_\infty,$$

$$(b-\mu)(\mu-a)-M_2 \le ||f||_p (b-a)^{2+\frac{1}{q}}[B(q+1,q+1)], \quad p>1, \ \frac{1}{p}+\frac{1}{q}=1.$$

Other estimations for M_2 from (2.2) and (3.1) are

$$m\frac{(b-a)^3}{6} \le (b-\mu)(\mu-a)-M_2 \le M\frac{(b-a)^3}{6}, \ m \le f \le M,$$

resulting in

$$M_2 \le (b-\mu)(\mu-a)-m\frac{(b-a)^3}{6}, \ m \le f \le M. \tag{3.7}$$

3.2. Bounds for the Third Central Moment M_3

From (2.3) and (3.2), the upper bound for M_3

$$M_3 \le (b-\mu)(\mu-a)^2 + [(b-\mu)-2(\mu-a)]M_2.$$

Further we obtain from (2.3) and (3.4),

$$M_3 \le (b-\mu)(\mu-a)(a+b-2\mu), \tag{3.8}$$

from (2.3) and (3.5),

$$M_3 \le \frac{1}{4}[(b-\mu)^3 + (b-\mu)(\mu-a)^2 - 2(\mu-a)^3], \tag{3.9}$$

and from (2.3) and (3.7),

$$M_3 \leq (b-\mu)(\mu-a)(a+b-2\mu) - \frac{m(b-a)^3(b+\mu-2a)}{6}. \tag{3.10}$$

3.3. Bounds for the Fourth Central Moment M_4

The upper bounds for M_4 from (2.4) and (3.2)

$$M_4 \leq (b-\mu)(\mu-a)^3 + 3(\mu-a)[(b-\mu)-(\mu-a)]M_2 + [(b-\mu)-3(\mu-a)]M_3.$$

Using (2.4), (3.4) and (3.8), we have

$$M_4 \leq (b-\mu)(\mu-a)[(b-a)^2 - 3(b-\mu)(\mu-a)], \tag{3.11}$$

from (2.4), (3.5) and (3.9),

$$M_4 \leq \frac{1}{4}\left[(b-\mu)^4 + 4(b-\mu)^2(\mu-a)^2 - 4(b-\mu)(\mu-a)^3 + 3(\mu-a)^4\right], \tag{3.12}$$

and from (2.4), (3.7) and (3.10),

$$M_4 \leq (b-\mu)(\mu-a)[(\mu-a)^2 + (a+b-2\mu)(a+b-4\mu) + 3(b-\mu)(a+b-2\mu)] - \frac{m(b-a)^3(a+b-2\mu)(b-2a-\mu)}{6}. \tag{3.13}$$

4. Results Based on the Grüss Type Inequality

We prove the following theorem based on the pre-Grüss inequality:

Theorem 4.1. *For the random variable X with distribution function $F : [a,b] \to [0,1]$,*

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right| \leq \frac{1}{2}(M-m)(b-a)^{r+s+1} \times \left[\frac{\Gamma(2r+1)\Gamma(2s+1)}{\Gamma(2r+2s+2)} - \left(\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right)^2\right]^{\frac{1}{2}}, \tag{4.1}$$

where $m \leq f \leq M$ a.e. on $[a,b]$ and $r, s \geq 0$.

Proof. We apply the following pre-Grüss inequality [4]:

$$\left|\int_a^b h(t)g(t)dt - \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)dt\right| \leq \frac{1}{2}(\phi-\gamma) \cdot \left[\frac{1}{b-a}\int_a^b g^2(t)dt - \left(\frac{1}{b-a}\int_a^b g(t)dt\right)^2\right]^{\frac{1}{2}}, \tag{4.2}$$

provided the mappings $h, g : [a,b] \to \mathbb{R}$ are measurable, all integrals involved exist and are finite and $\gamma \leq h \leq \phi$ a.e. on $[a,b]$.

Let $h(t) = f(t),\ g(t) = (b-t)^r (t-a)^s$ in (4.2). Then

$$\begin{aligned}(4.3)\quad & \left| \int_a^b (b-t)^r (t-a)^s f(t)dt \right. \\ & \qquad \left. - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b (b-t)^r (t-a)^s dt \right| \\ & \qquad \leq \frac{1}{2}(M-m) \cdot \left[\frac{1}{b-a} \int_a^b \{(b-t)^r (t-a)^s\}^2 \, dt \right. \\ & \qquad\qquad \left. - \left(\frac{1}{b-a} \int_a^b (b-t)^r (t-a)^s dt \right)^2 \right]^{\frac{1}{2}},\end{aligned}$$

where $m \leq f \leq M$ a.e. on $[a.b]$.

On substituting from (3.2) into (4.3), we prove the theorem. □

Corollary 4.2. *For $r = s = 1$ in (4.2),*

$$\left| \int_a^b (b-t)(t-a) f(t)dt - \frac{(b-a)^2}{6} \right| \leq \frac{(M-m)(b-a)^3}{12\sqrt{5}},$$

a result (2.7) in Theorem 1 by Barnett and Dragomir [1].

We have the following lemma based on the pre-Grüss inequality:

Lemma 4.3. *For the random variable X with distribution function $F : [a,b] \to$*

$[0,1]$,

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right| \leq \frac{1}{2}(M-m)\left[(b-a)\int_a^b f^2(t)dt - 1\right]^{\frac{1}{2}}, \tag{4.4}$$

where $m \leq f \leq M$ *a.e. on* $[a,b]$ *and* $r,s \geq 0$.

Proof. We choose $h(t) = (b-t)^r (t-a)^s,\ \ g(t) = f(t)$ in the pre-Grüss inequality (4.2) to prove this lemma. □

We now prove the following theorems based on Lemma 4.3:

Theorem 4.4. *For the random variable* X *with distribution function* $F : [a,b] \to [0,1]$,

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right| \leq \frac{1}{4}(b-a)(M-m)^2, \tag{4.5}$$

where $m \leq f \leq M$ *a.e. on* $[a,b]$ *and* $r,s \geq 0$.

Proof. Barnett and Dragomir [3] established the following identity:

$$\frac{1}{b-a}\int_a^b f(t)g(t)dt = p + \left(\frac{1}{b-a}\right)^2 \cdot \int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt, \tag{4.6}$$

where
$$|p| \le \frac{1}{4}(\Gamma - \gamma)(\Phi - \phi), \quad \text{and} \quad \Gamma < f < \gamma, \Phi < g < \phi.$$
By taking $g = f$ in (4.6), we get

$$(4.7) \quad \frac{1}{b-a}\int_a^b f^2(t)dt = p + \left(\frac{1}{b-a}\right)^2, \text{ where } |p| \le \frac{1}{4}(M-m), \quad M < f < m.$$

Thus, (4.4) and (4.7) prove the theorem. □

Another inequality based on a result from Barnett and Dragomir [3] follows:

Theorem 4.5. *For the random variable X with distribution function $F : [a,b] \to [0,1]$,*

$$(4.8) \quad \left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s} \cdot \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right| \le \frac{1}{4}M(M-m)(b-a),$$

where $m \le f \le M$ a.e. on $[a,b]$ and $r, s \ge 0$.

Proof. Barnett and Dragomir [3] have established the following inequality:

$$(4.9) \quad \left|\frac{1}{b-a}\int_a^b f^n(t)dt - \left(\frac{1}{b-a}\right)^n\right| \le \frac{\Gamma^2}{4(b-a)^{n-2}}\left[\frac{\Gamma^{n-1}\cdot(b-a)^{n-1} - 1}{\Gamma\cdot(b-a) - 1}\right],$$

where $\gamma < f < \Gamma$.

From (4.9), we get

$$\left[\left|\frac{1}{b-a}\int_a^b f^2(t)dt - \left(\frac{1}{b-a}\right)^2\right|\right]^{\frac{1}{2}} \leq \frac{M}{2}, m \leq f \leq M.$$

and substituting in (4.4) proves the theorem. □

5. Results Based on the Hölder's Integral Inequality

We consider the Hölder's integral inequality [4] and for $t \in [a,b]$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$,

$$\left|\int_a^t (t-u)^n\, f^{(n+1)}(u)du\right| \le \left(\int_a^t |f^{(n+1)}(u)|du\right)^{\frac{1}{p}} \cdot \left(\int_a^t (t-u)^{nq}du\right)^{\frac{1}{q}} \le ||f^{(n+1)}||_p \cdot \left[\frac{(t-a)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}. \tag{5.1}$$

On applying (5.1),we have the theorem:

Theorem 5.1. *For the random variable X with distribution function $F:[a,b]\to[0,1]$, suppose that the density function $f:[a,b]$ is $n-$ times differentiable and $f^{(n)}$ $(n\ge 0)$ is absolutely continuous on $[a,b]$. Then,*

$$\left|\int_a^b (t-a)^r(b-t)^s f(t)dt - \sum_{k=0}^{n}(b-a)^{r+s+k+1}\cdot\frac{\Gamma(s+1)\Gamma(r+k+1)}{\Gamma(r+s+k+2)}\right| \tag{5.2}$$

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$$\leq \frac{1}{n!}\cdot\begin{cases} \dfrac{||f^{(n+1)}||_\infty}{n+1}\cdot(b-a)^{r+s+n+2}\cdot\dfrac{\Gamma(r+n+2)\Gamma(s+1)}{\Gamma(r+s+n+3)}, \\ \qquad\qquad\qquad\qquad \textit{if } f^{(n+1)}\in L_\infty[a,b], \\ \dfrac{||f^{(n+1)}||_p}{(nq+1)^{1/q}}\cdot(b-a)^{r+s+n+\frac{1}{q}+1}\cdot\dfrac{\Gamma(r+n+\frac{1}{q}+1)\Gamma(s+1)}{\Gamma(r+s+n+\frac{1}{q}+2)}, \\ \qquad\qquad\qquad\qquad \textit{if } f^{(n+1)}\in L_p[a,b]\ ,\ p>1, \\ ||f^{(n+1)}||_1\cdot(b-a)^{r+s+n+1}\cdot\dfrac{\Gamma(r+n+1)\Gamma(s+1)}{\Gamma(r+s+n+2)}, \\ \qquad\qquad\qquad\qquad \textit{if } f^{(n+1)}\in L_1[a,b], \end{cases}$$

where $||.||_p$ $(1\leq p\leq\infty)$ *are the Lebesgue norms on* $[a,b]$, *i.e.,*

$$||g||_\infty := ess\ \sup_{t\in[a,b]}|g(t), \textit{ and } \ ||g||_p := \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}},\quad (p\geq 1).$$

Proof. Using the Taylor's expansion of f about a :

$$f(t)=\sum_{k=0}^{n}\frac{(t-a)^k}{k!}f^k(a)+\frac{1}{n!}\int_a^t (t-u)^n f^{(n+1)}(u)du,\ \ t\in[a,b],$$

we have

$$\int_a^b (t-a)^r(b-t)^s f(t)dt=\sum_{k=0}^{n}\left[\int_a^b (t-a)^{r+k}(b-t)^s dt\cdot\frac{f^k(a)}{k!}\right] + \left[\frac{1}{n!}\int_a^b (t-a)^r(b-t)^s\left(\int_a^t (t-u)^n\ f^{(n+1)}(u)du\right)dt\right]. \tag{5.3}$$

Applying the transformation $t = (1-x)a + xb$, we have

$$\int_a^b (t-a)^{r+k}(b-t)^s dt = (b-a)^{r+s+k+1} \cdot \frac{\Gamma(s+1)\Gamma(r+k+1)}{\Gamma(r+s+k+2)}. \tag{5.4}$$

For $t \in [a,b]$,it may be seen that

$$\begin{aligned} \left| \int_a^t (t-u)^n f^{(n+1)}(u) du \right| &\leq \int_a^t |(t-u)^n| | f^{(n+1)}(u)| du \\ &\leq \sup_{u \in [a,b]} |f^{(n+1)}(u)| \cdot \int_a^t (t-u)^n du \\ &\leq ||f^{(n+1)}||_\infty \cdot \frac{(t-a)^{n+1}}{n+1}. \end{aligned} \tag{5.5}$$

Further, for $t \in [a,b]$,

$$\begin{aligned} \left| \int_a^t (t-u)^n f^{(n+1)}(u) du \right| &\leq \int_a^t (t-u)^n |f^{(n+1)}(u)| du \\ &\leq (t-a)^n \int_a^t |f^{(n+1)}(u)| du \\ &\leq ||f^{(n+1)}|| \cdot (t-a)^n. \end{aligned} \tag{5.6}$$

Let

$$M(a,b) := \frac{1}{n!} \int_a^b (t-a)^r (b-t)^s \left(\int_a^t (t-u)^n f^{(n+1)}(u) du \right) dt. \tag{5.7}$$

Then (5.1) and (5.5) to (5.7) result in

$$M(a,b) \leq \frac{1}{n!}\cdot\begin{cases} \frac{\|f^{(n+1)}\|_\infty}{n+1}\cdot\int_a^b (t-a)^{r+n+1}(b-t)^s dt, & \text{if } f^{(n+1)}\in L_\infty[a,b], \\ \frac{\|f^{(n+1)}\|_p}{(nq+1)^{1/q}}\cdot\int_a^b (t-a)^{r+n+\frac{1}{q}}(b-t)^s dt, & \text{if } f^{(n+1)}\in L_p[a,b]\,,\ p>1, \\ \|f^{(n+1)}\|_1\cdot\int_a^b (t-a)^{r+n}(b-t)^s dt, & \text{if } f^{(n+1)}\in L_1[a,b]. \end{cases} \tag{5.8}$$

Using (5.3), (5.4) and (5.8), we prove the theorem. □

Corollary 5.2. *Considering $r=s=1$, the inequality (5.8) leads to*

$$M(a,b) \leq \frac{1}{n!}\begin{cases} \frac{\|f^{(n+1)}\|_\infty}{(n+1)}\cdot\frac{(b-a)^{n+4}}{(n+3)(n+4)}, & \textit{if } f^{(n+1)}\in L_\infty[a,b], \\ \frac{\|f^{(n+1)}\|_p}{(nq+1)^{\frac{1}{q}}}\cdot\frac{(b-a)^{n+\frac{1}{q}+3}}{\left(n+\frac{1}{q}+2\right)\left(n+\frac{1}{q}+3\right)}, & \textit{if } f^{(n+1)}\in L_p[a,b]\,,\ p>1, \\ \|f^{(n+1)}\|_1\cdot\frac{(b-a)^{n+3}}{(n+2)(n+3)}, & \textit{if } f^{(n+1)}\in L_1[a,b], \end{cases},$$

which is Theorem 3 of Barnett and Dragomir [1].

6. Application to the Truncated Exponential Distribution

The truncated exponential distribution arises frequently in applications particularly in insurance contracts with caps and deductible and in the field of life-testing. A random variable X with distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } 0 \le x < c, \\ 1 & \text{for } x \ge c, \end{cases}$$

is a truncated exponential distribution with parameters λ and c.

The density function for X :

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } 0 \le x < c \\ 0 & \text{for } x \ge c \end{cases} + e^{-\lambda c} \cdot \delta_c(x),$$

where δ_c is the delta function at $x = c$. This distribution is therefore mixed with a continuous distribution $f(x) = \lambda e^{-\lambda x}$ on the interval $0 \le x < c$ and a point mass of size $e^{-\lambda c}$ at $x = c$.

The moment generating function for the random variable X:

$$\begin{aligned} M_X(t) &= \int_0^c e^{tx} \cdot \lambda e^{-\lambda x} dx + e^{tc} \cdot e^{-\lambda c} \\ &= \begin{cases} \dfrac{\lambda - t e^{-c(\lambda - t)}}{\lambda - t}, & \text{for } t \ne \lambda, \\ \lambda c + 1, & \text{for } t = \lambda. \end{cases} \end{aligned}$$

For further calculations in what follows, we assume $t \neq \lambda$. From the moment generating function $M_X(t)$, we have:

$$\begin{aligned} E(X) &= \frac{1-e^{-\lambda c}}{\lambda}, \\ E(X^2) &= \frac{2[1-(1+\lambda c)e^{-\lambda c}]}{\lambda^2}, \\ E(X^3) &= \frac{3[2-(2+2\lambda c+\lambda^2 c^2)e^{-\lambda c}]}{\lambda^3}, \\ E(X^4) &= \frac{4[6-(6+6\lambda c+3\lambda^2 c^2+\lambda^3 c^3)e^{-\lambda c}]}{\lambda^4}. \end{aligned}$$

The higher order central moments are:

$$M_k = \sum_{i=0}^{k} \binom{k}{i} E(X^i) \cdot \mu^{k-i}, \text{ for } k = 2, 3, 4, \ldots,$$

in particular,

$$\begin{aligned} M_2 &= \frac{1-2\lambda c e^{-\lambda c}-e^{-2\lambda c}}{\lambda^2}, \\ M_3 &= \frac{16-3e^{-\lambda c}(10+4\lambda c+\lambda^2 c^2)+6e^{-2\lambda c}(3+\lambda c)-4e^{-3\lambda c}}{\lambda^3}, \\ M_4 &= \frac{65-4e^{-\lambda c}(32+15\lambda c+6\lambda^2 c^2+\lambda^3 c^3)}{\lambda^4} \\ &\quad + \frac{3e^{-2\lambda c}(30+16\lambda c+4\lambda^2 c^2)-4e^{-3\lambda c}(8+3\lambda c)+5e^{-4\lambda c}}{\lambda^4}. \end{aligned}$$

Using the moment-estimation inequality (3.6), the upper bound for M_2, in terms of the parameters λ and c of the distribution:

$$\hat{M}_2 \leq \frac{(1-e^{-\lambda c})(\lambda c - 1 + e^{-\lambda c})}{\lambda^2}.$$

The upper bounds for M_3 using (3.8)

$$\hat{M}_3 \leq \frac{(2-3\lambda c+\lambda^2c^2) - e^{-\lambda c}(6-6\lambda c+\lambda^2c^2) + 3e^{-2\lambda c}(2-\lambda c) - 2e^{-3\lambda c}}{\lambda^3},$$

and using (3.9)

$$\hat{M}_3 \leq \frac{(-3+4\lambda c-3\lambda^2c^2+\lambda^3c^3)}{4\lambda^3} + \frac{e^{-\lambda c}(9-8\lambda c+3\lambda^2c^2) - e^{-2\lambda c}(9-4\lambda c) + 3e^{-3\lambda c}}{4\lambda^3}.$$

The upper bounds for M_4 using (3.11)

$$\hat{M}_4 \leq \frac{(-3+6\lambda c-4\lambda^2c^2+\lambda^3c^3) + e^{-\lambda c}(12-18\lambda c+8\lambda^2c^2-\lambda^3c^3)}{\lambda^4} - \frac{2e^{-2\lambda c}(9+9\lambda c+2\lambda^2c^2) - 6e^{-3\lambda c}(2-\lambda c) + 3e^{-4\lambda c}}{\lambda^4},$$

and from (3.12),

$$\hat{M}_4 \leq \frac{(12-16\lambda c+103\lambda^2c^2-4\lambda^3c^3+\lambda^4c^4)}{4\lambda^4}$$

$$-\frac{4e^{-\lambda c}(12-12\lambda c+5\lambda^2c^2-\lambda^3c^3)}{4\lambda^4}$$
$$+\frac{2e^{-2\lambda c}(36+24\lambda c+5\lambda^2c^2)-16e^{-3\lambda c}(3-\lambda c)+12e^{-4\lambda c}}{4\lambda^4}.$$

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