

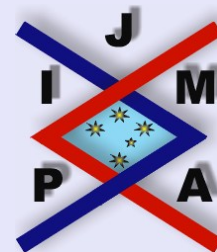
# Journal of Inequalities in Pure and Applied Mathematics

## ON MULTIVARIATE OSTROWSKI TYPE INEQUALITIES

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

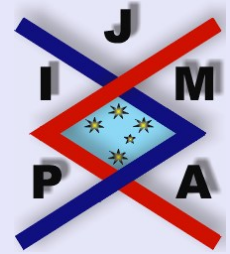
In the present paper we establish new multivariate Ostrowski type inequalities by using fairly elementary analysis.

*2000 Mathematics Subject Classification:* 26D15, 26D20.

*Key words:* Multivariate, Ostrowski type inequalities, Many independent variables,  $n$ -fold integral.

## Contents

1	Introduction .....	3
2	Statement of Results .....	4
3	Proof of Theorem 2.1 .....	7
4	Proof of Theorem 2.2 .....	9
	References	



---

### On Multivariate Ostrowski Type Inequalities

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 12

# 1. Introduction

The following inequality is well known in the literature as Ostrowski's integral inequality (see [5, p. 469]).

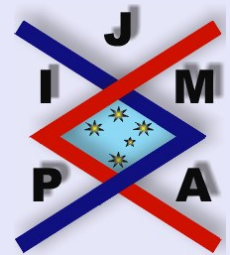
Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all  $x \in [a, b]$ .

Many generalisations, extensions and variants of this inequality have appeared in the literature, see [1] – [7] and the references given therein.

The main aim of this paper is to establish new inequalities similar to that of Ostrowski's inequality involving functions of many independent variables and their first order partial derivatives. The analysis used in the proof is elementary and our results provide new estimates on these types of inequalities.



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On Multivariate Ostrowski Type Inequalities

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 12

## 2. Statement of Results

In what follows,  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}^n$  the  $n$ -dimensional Euclidean space. Let  $D = \{(x_1, \dots, x_n) : a_i < x_i < b_i \ (i = 1, \dots, n)\}$  and  $\bar{D}$  be the closure of  $D$ . For a function  $u(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote the first order partial derivatives by  $\frac{\partial u(x)}{\partial x_i}$  ( $i = 1, \dots, n$ ) and  $\int_D u(x) dx$  the  $n$ -fold integral  $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \dots, x_n) dx_1 \dots dx_n$ .

Our main results are established in the following theorems.

**Theorem 2.1.** *Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous functions on  $\bar{D}$  and differentiable on  $D$  whose derivatives  $\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$  are bounded, i.e.,*

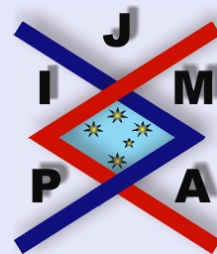
$$\left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} = \sup_{x \in D} \left| \frac{\partial f(x)}{\partial x_i} \right| < \infty,$$

$$\left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} = \sup_{x \in D} \left| \frac{\partial g(x)}{\partial x_i} \right| < \infty.$$

*Let the function  $w(x)$  be defined, nonnegative, integrable for every  $x \in D$  and  $\int_D w(y) dy > 0$ . Then for every  $x \in \bar{D}$ ,*

$$(2.1) \quad \left| f(x)g(x) - \frac{1}{2M}g(x) \int_D f(y) dy - \frac{1}{2M}f(x) \int_D g(y) dy \right|$$

$$\leq \frac{1}{2M} \sum_{i=1}^n \left[ |g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} \right] E_i(x),$$



On Multivariate Ostrowski Type  
Inequalities

B.G. Pachpatte

Title Page

Contents

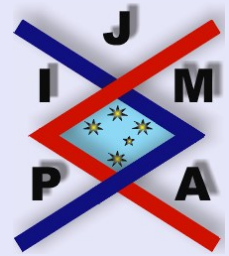


Go Back

Close

Quit

Page 4 of 12



Title Page

Contents



Go Back

Close

Quit

Page 5 of 12

$$(2.2) \quad \left| f(x)g(x) - \left[ \frac{g(x) \int_D w(y) f(y) dy + f(x) \int_D w(y) g(y) dy}{2 \int_D w(y) dy} \right] \right| \\ \leq \frac{\int_D w(y) \sum_{i=1}^n \left[ |g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} \right] |x_i - y_i| dy}{2 \int_D w(y) dy},$$

where

$$M = \text{mes } D = \prod_{i=1}^n (b_i - a_i), \quad dy = dy_1 \dots dy_n \quad \text{and} \quad E_i(x) = \int_D |x_i - y_i| dy.$$

**Remark 2.1.** If we take  $g(x) = 1$  and hence  $\frac{\partial g}{\partial x_i} = 0$  in Theorem 2.1, then the inequalities (2.1) and (2.2) reduces to the inequalities established by Milovanović in [3, Theorems 2 and 3] which in turn are the further generalisations of the well known Ostrowski's inequality.

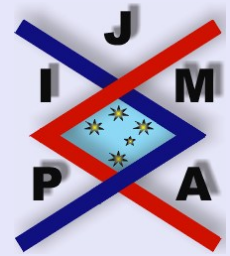
**Theorem 2.2.** Let  $f, g, \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$  be as in Theorem 2.1. Then for every  $x \in \bar{D}$ ,

$$(2.3) \quad \left| f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) \right. \\ \left. - g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M} \int_D f(y)g(y) dy \right| \\ \leq \frac{1}{M} \int_D \left[ \left( \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right) \left( \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right) \right] dy,$$

$$(2.4) \quad \left| f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) - g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M^2} \left( \int_D f(y) dy \right) \left( \int_D g(y) dy \right) \right| \leq \frac{1}{M^2} \left( \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} E_i(x) \right) \left( \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} E_i(x) \right),$$

where  $M$ ,  $dy$  and  $E_i(x)$  are as defined in Theorem 2.1.

**Remark 2.2.** We note that in [1] Anastassiou has used a slightly different technique to establish multivariate Ostrowski type inequalities. However, the inequalities established in (2.3) and (2.4) are different from those given in [1] and our proofs are extremely simple. For an  $n$ -dimensional version of Ostrowski's inequality for mappings of Hölder type, see [2].




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### On Multivariate Ostrowski Type Inequalities

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 6 of 12

### 3. Proof of Theorem 2.1

Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  ( $x \in \bar{D}$ ,  $y \in D$ ). From the  $n$ -dimensional version of the mean value theorem, we have (see [8, p. 174] or [4, p. 121])

$$(3.1) \quad f(x) - f(y) = \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i),$$

$$(3.2) \quad g(x) - g(y) = \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i),$$

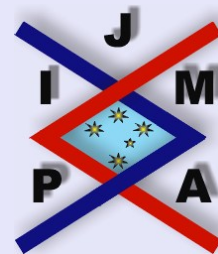
where  $c = (y_1 + \alpha(x_1 - y_1), \dots, y_n + \alpha(x_n - y_n))$  ( $0 < \alpha < 1$ ).

Multiplying both sides of (3.1) and (3.2) by  $g(x)$  and  $f(x)$  respectively and adding, we get

$$(3.3) \quad 2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ = g(x) \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) + f(x) \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i).$$

Integrating both sides of (3.3) with respect to  $y$  over  $D$ , using the fact that  $\text{mes } D > 0$  and rewriting, we have

$$(3.4) \quad f(x)g(x) - \frac{1}{2M}g(x) \int_D f(y) dy - \frac{1}{2M}f(x) \int_D g(y) dy$$



On Multivariate Ostrowski Type  
Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 7 of 12

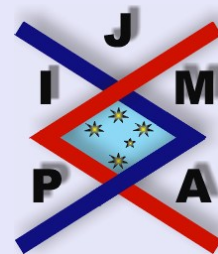
$$= \frac{1}{2M} \left[ g(x) \int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) dy + f(x) \int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) dy \right].$$

From (3.4) and using the properties of modulus we have

$$\begin{aligned} & \left| f(x)g(x) - \frac{1}{2M}g(x) \int_D f(y) dy - \frac{1}{2M}f(x) \int_D g(y) dy \right| \\ & \leq \frac{1}{2M} \left[ |g(x)| \int_D \sum_{i=1}^n \left| \frac{\partial f(c)}{\partial x_i} \right| |x_i - y_i| dy \right. \\ & \quad \left. + |f(x)| \int_D \sum_{i=1}^n \left| \frac{\partial g(c)}{\partial x_i} \right| |x_i - y_i| dy \right] \\ & \leq \frac{1}{2M} \sum_{i=1}^n \left[ |g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} \right] E_i(x). \end{aligned}$$

The proof of the inequality (2.1) is complete.

Multiplying both sides of (3.3) by  $w(y)$  and integrating the resulting identity with respect to  $y$  on  $D$  and following the proof of inequality (2.1), we get the desired inequality in (2.2).




---

**On Multivariate Ostrowski Type Inequalities**

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 8 of 12



## 4. Proof of Theorem 2.2

From the hypotheses, as in the proof of Theorem 2.1, the identities (3.1) and (3.2) hold. Multiplying the left and right sides of (3.1) and (3.2) we get

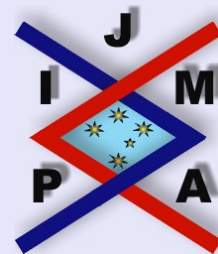
$$(4.1) \quad f(x)g(x) - f(x)g(y) - g(x)f(y) + f(y)g(y) \\ = \left[ \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) \right] \left[ \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) \right].$$

Integrating both sides of (4.1) with respect to  $y$  on  $D$  and rewriting, we have

$$(4.2) \quad f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) \\ - g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M} \int_D f(y)g(y) dy \\ = \frac{1}{M} \int_D \left[ \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) \right] \left[ \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) \right] dy.$$

From (4.2) and using the properties of the modulus, we have

$$\left| f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) \right. \\ \left. - g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M} \int_D f(y)g(y) dy \right|$$



---

On Multivariate Ostrowski Type  
Inequalities

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 9 of 12

$$\begin{aligned} &\leq \frac{1}{M} \int_D \left[ \sum_{i=1}^n \left| \frac{\partial f(c)}{\partial x_i} \right| |x_i - y_i| \right] \left[ \sum_{i=1}^n \left| \frac{\partial g(c)}{\partial x_i} \right| |x_i - y_i| \right] dy \\ &\leq \frac{1}{M} \int_D \left[ \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right] \left[ \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right] dy, \end{aligned}$$

which is the required inequality in (2.3).

Integrating both sides of (3.1) and (3.2) with respect to  $y$  over  $D$  and rewriting, we get

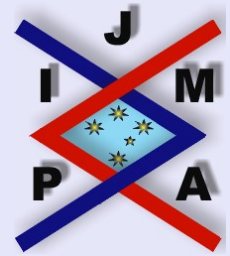
$$(4.3) \quad f(x) - \frac{1}{M} \int_D f(y) dy = \frac{1}{M} \int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) dy,$$

and

$$(4.4) \quad g(x) - \frac{1}{M} \int_D g(y) dy = \frac{1}{M} \int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) dy,$$

respectively. Multiplying the left and right sides of (4.3) and (4.4) we get

$$\begin{aligned} (4.5) \quad &f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) \\ &- g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M^2} \left( \int_D f(y) dy \right) \left( \int_D g(y) dy \right) \\ &= \frac{1}{M^2} \left( \int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) dy \right) \left( \int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) dy \right). \end{aligned}$$



### On Multivariate Ostrowski Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

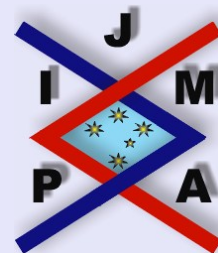
Quit

Page 10 of 12

From (4.5) and using the properties of the modulus we have

$$\begin{aligned}
 & \left| f(x)g(x) - f(x) \left( \frac{1}{M} \int_D g(y) dy \right) \right. \\
 & \quad \left. - g(x) \left( \frac{1}{M} \int_D f(y) dy \right) + \frac{1}{M^2} \left( \int_D f(y) dy \right) \left( \int_D g(y) dy \right) \right| \\
 & \leq \frac{1}{M^2} \left( \int_D \sum_{i=1}^n \left| \frac{\partial f(c)}{\partial x_i} \right| |x_i - y_i| dy \right) \left( \int_D \left| \frac{\partial g(c)}{\partial x_i} \right| |x_i - y_i| dy \right) \\
 & \leq \frac{1}{M^2} \left( \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} E_i(x) \right) \left( \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} E_i(x) \right).
 \end{aligned}$$

This is the desired inequality in (2.4) and the proof is complete.



On Multivariate Ostrowski Type  
Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

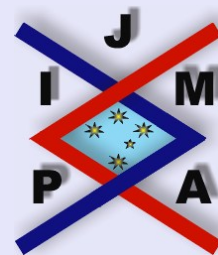
Close

Quit

Page 11 of 12

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---

### On Multivariate Ostrowski Type Inequalities

B.G. Pachpatte

---

Title Page

Contents



Go Back

Close

Quit

Page 12 of 12