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FURTHER DEVELOPMENT OF AN OPEN PROBLEM

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ABSTRACT. In this paper, we generalize an open problem posed by Ngô et al. in [Notes on an Integral Inequality, JIPAM, **7**(4) (2006), Art.120] and give some answers which extend the results of Boukerrioua-Guezane-Lakoud [On an open question regarding an integral inequality, JIPAM, **8**(3) (2007), Art. 77.] and Liu-Li-Dong [On an open problem concerning an integral inequality, JIPAM, **8** (3) (2007), Art. 74.].

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1. INTRODUCTION

In [4], the following inequality is found.

Theorem A. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)\mathrm{d}t \geq \int_x^1 t\mathrm{d}t, \qquad \forall\, x \in [0,1], \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)\mathrm{d}x \geq \int_0^1 x^\alpha f(x)\mathrm{d}x, \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)\mathrm{d}x \geq \int_0^1 x f^\alpha(x)\mathrm{d}x, \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

The authors next proposed the following open problem:

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Open Problem 1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)\mathrm{d}t \geq \int_x^1 t\mathrm{d}t, \qquad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x) \geq \int_0^1 x^\alpha f^\beta(x)\mathrm{d}x \tag{1.5}$$

hold for α and β?

Several answers and extension results have been given to this open problem [1, 3]. In the present paper, we obtain a generalization of the above Open Problem 1 and provide some resolutions to it. Here, and in what follows, we use X to denote a non-negative random variable (r.v.) on $[0,\infty)$ with probability density function $p(x)$. $\mathbf{E}(X)$ denotes the mathematical expectation of X and $\mathbf{1}_A := \mathbf{1}_A(X)$ denotes the indicator function of the event A. Let $A_t = [t,\infty)$. We now consider the following generalization of Open Problem 1.

Open Problem 2. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$. Under what conditions does the inequality*

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^\alpha f^\beta(X)) \tag{1.6}$$

hold for α and β?

Remark 1. Let X possess a uniform distribution on the support interval $[0,1]$, i.e., the probability density function of X is equal to 1, $x \in [0,1]$ and zero elsewhere, then the above open problem becomes Open Problem 1.

For convenience, we assume that all the necessary functions in the following are integrable.

2. MAIN RESULTS

Theorem 2.1. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ satisfying*

$$\mathbf{E}(f^\beta(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X^\beta \mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.1}$$

Then

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^\alpha f^\beta(X)) \tag{2.2}$$

holds for every positive real number α and β.

Proof. By Fubini's Theorem and (2.1), we have

$$\begin{aligned}\mathbf{E}(X^\alpha f^\beta(X)) &= \int_0^\infty x^\alpha f^\beta(x)p(x)\mathrm{d}x \\ &= \frac{1}{\alpha}\int_0^\infty \left(\int_0^x t^{\alpha-1}\mathrm{d}t\right) f^\beta(x)p(x)\mathrm{d}x \\ &= \frac{1}{\alpha}\int_0^\infty t^{\alpha-1}\left(\int_t^\infty f^\beta(x)p(x)\mathrm{d}x\right)\mathrm{d}t \\ &= \frac{1}{\alpha}\int_0^\infty t^{\alpha-1}\mathbf{E}(f^\beta(X)\mathbf{1}_{A_t})\mathrm{d}t\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&\geq \frac{1}{\alpha}\int_0^\infty t^{\alpha-1}\mathbf{E}(X^\beta \mathbf{1}_{A_t})\mathrm{d}t \\
&= \frac{1}{\alpha}\int_0^\infty t^{\alpha-1}\left(\int_t^\infty x^\beta p(x)\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{\alpha}\int_0^\infty \left(\int_0^x t^{\alpha-1}\mathrm{d}t\right) x^\beta p(x)\mathrm{d}x \\
&= \int_0^\infty x^{\beta+\alpha}p(x)\mathrm{d}x = \mathbf{E}(X^{\beta+\alpha}).
\end{aligned}
$$

Using Cauchy's inequality, we have

$$
\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(X) + \frac{\alpha}{\alpha+\beta}X^{\alpha+\beta} \geq X^\alpha f^\beta(X), \tag{2.4}
$$

which, by (2.3), yields

$$
\frac{\beta}{\alpha+\beta}\mathbf{E}(f^{\alpha+\beta}(X)) + \frac{\alpha}{\alpha+\beta}\mathbf{E}(X^{\alpha+\beta}) \geq \mathbf{E}(X^\alpha f^\beta(X)) \geq \mathbf{E}(X^{\beta+\alpha}). \tag{2.5}
$$

□

Remark 2. If we assume that X possesses a uniform distribution on the support interval $[0,1]$ (or $[0,b]$), then the above theorem is Theorem 2.1 (or Theorem 2.4) of Liu-Li-Dong in [3].

Theorem 2.2. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ satisfying*

$$
\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.6}
$$

Then

$$
\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^\beta f^\alpha(X)) \tag{2.7}
$$

holds for every positive real number $\alpha \geq 1$ and β.

Proof. By Fubini's Theorem and (2.6), we have

$$
\begin{aligned}
\mathbf{E}(X^{\alpha+1}f(X)) &= \int_0^\infty x^{\alpha+1}f(x)p(x)\mathrm{d}x \\
&= \frac{1}{\alpha+1}\int_0^\infty \left(\int_0^x t^\alpha \mathrm{d}t\right) f(x)p(x)\mathrm{d}x \\
&= \frac{1}{\alpha+1}\int_0^\infty t^\alpha \left(\int_t^\infty f(x)p(x)\mathrm{d}x\right)\mathrm{d}t \\
&\geq \frac{1}{\alpha+1}\int_0^\infty t^\alpha \left(\int_t^\infty xp(x)\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{\alpha+1}\int_0^\infty xp(x)\left(\int_0^x t^\alpha \mathrm{d}t\right)\mathrm{d}x = \mathbf{E}(X^{\alpha+2}).
\end{aligned} \tag{2.8}
$$

Applying Cauchy's inequality, we get

$$
\frac{1}{\alpha}f^\alpha(X) + \frac{\alpha-1}{\alpha}X^\alpha \geq f(X)X^{\alpha-1}. \tag{2.9}
$$

Multiplying both sides of (2.9) by X^β, we have

$$
\frac{1}{\alpha}\mathbf{E}(f^\alpha(X)X^\beta) + \frac{\alpha-1}{\alpha}\mathbf{E}(X^{\alpha+\beta}) \geq \mathbf{E}(f(X)X^{\alpha+\beta-1}) \geq \mathbf{E}(X^{\alpha+\beta}), \tag{2.10}
$$

which implies

$$\mathbf{E}(f^{\alpha}(X)X^{\beta}) \geq \mathbf{E}(X^{\alpha+\beta}). \tag{2.11}$$

The remainder of the proof is similar to that of Theorem 2.1. □

Remark 3. If we assume that X possesses a uniform distribution on the support interval $[0,1]$ then the above theorem is Theorem 2.3 of Boukerrioua and Guezane-Lakoud in [1].

Next we consider the case "$\alpha > 0, \beta > 2$" by using the ideas of Dragomir-Ngô in [2].

Lemma 2.3 ([2])**.** *Let $f : [a,b] \to [0,\infty)$ be a continuous function and $g : [a,b] \to [0,\infty)$ be non-decreasing, differentiable on (a,b) satisfying*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt.$$

Then

$$\int_x^b f^{\beta}(t)dt \geq \int_x^b g^{\beta}(t)dt$$

holds for $\beta > 1$.

Lemma 2.4. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ with $f'(x) \geq 0$ on $(0,\infty)$ and satisfying*

$$\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.12}$$

Then

$$\mathbf{E}(f^{\beta}(X)\mathbf{1}_{A_t})) \geq \mathbf{E}(X^{\beta}\mathbf{1}_{A_t}) \tag{2.13}$$

holds for every positive real number $\beta \geq 1$.

Proof. The proof is a direct extension of Theorem 3 in [2]. □

Theorem 2.5. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ with $f'(x) \geq 0$ on $(0,\infty)$ and satisfying*

$$\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.14}$$

In addition, for every positive real number $\alpha > 0, \beta > 2$ satisfying

$$\lim_{x\to\infty} f^{\alpha}(x)x^{\beta-1}\mathbf{E}[(f(X)-X)\mathbf{1}_{A_x}] = 0, \tag{2.15}$$

and

$$\lim_{x\to\infty} f^{\alpha}(x)x\mathbf{E}[(f^{\beta-1}(X)-X^{\beta-1})\mathbf{1}_{A_x}] = 0, \tag{2.16}$$

then

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^{\beta}f^{\alpha}(X)). \tag{2.17}$$

Proof. It is obvious that

$$(f(x)-x)(f^{\beta-1}(x)-x^{\beta-1}) \geq 0,$$

which implies that

$$f^{\beta+\alpha}(x) \geq x^{\beta-1}f^{1+\alpha}(x) + xf^{\beta-1+\alpha}(x) - x^{\beta}f^{\alpha}(x). \tag{2.18}$$

Integrating by parts and using (2.14) and (2.15), we have

$$\begin{aligned}
&\mathbf{E}(f^{\alpha}(X)X^{\beta-1}(f(X)-X))\\
&=\int_0^{\infty} f^{\alpha}(x)x^{\beta-1}(f(x)-x)p(x)dx\\
&=-\int_0^{\infty} f^{\alpha}(x)x^{\beta-1}\mathrm{d}\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=-f^{\alpha}(x)x^{\beta-1}\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)\Big|_0^{\infty}\\
&\qquad+\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\mathbf{E}\left((f(X)-X)\mathbf{1}_{A_x}\right)\mathrm{d}x\ge 0,
\end{aligned}$$

which yields

$$\mathbf{E}(f^{\alpha+1}(X)X^{\beta-1})\ge\mathbf{E}(f^{\alpha}(X)X^{\beta}). \tag{2.19}$$

Furthermore, by Lemma 2.4 and the condition (2.16), we have

$$\begin{aligned}
&\mathbf{E}(f^{\alpha}(X)X(f^{\beta-1}(X)-X^{\beta-1}))\\
&\quad=\int_0^{\infty} f^{\alpha}(x)x(f^{\beta-1}(x)-x^{\beta-1})p(x)dx\\
&\quad=-\int_0^{\infty} f^{\alpha}(x)x\mathrm{d}\left(\int_x^{\infty}(f^{\beta-1}(t)-t^{\beta-1})p(t)\mathrm{d}t\right)dx\\
&\quad=-f^{\alpha}(x)x\left(\int_x^{\infty}(f^{\beta-1}(t)-t^{\beta-1})p(t)\mathrm{d}t\right)\Big|_0^{\infty}\\
&\qquad\quad+\int_0^{\infty}(\alpha xf'(x)f^{\alpha-1}(x)+f^{\alpha}(x))\left(\int_x^{\infty}(f^{\beta-1}(t)-t^{\beta-1})p(t)\mathrm{d}t\right)dx\\
&\quad=\int_0^{\infty}(\alpha xf'(x)f^{\alpha-1}(x)+f^{\alpha}(x))\mathbf{E}\left((f^{\beta-1}(X)-X^{\beta-1})\mathbf{1}_{A_x}\right)\mathrm{d}x\ge 0,
\end{aligned}$$

which yields

$$\mathbf{E}(f^{\alpha+\beta-1}(X)X)\ge\mathbf{E}(f^{\alpha}(X)X^{\beta}). \tag{2.20}$$

From (2.18)-(2.20), inequality (2.17) holds. □

3. Further Discussion

Let $g(x)\ge 0$, $0<\int_0^{\infty}g(x)\mathrm{d}x<\infty$. If $p(x):=\frac{g(x)}{\int_0^{\infty}g(x)\mathrm{d}x}$, then it is easy to check that $p(x)$ is a probability density function on the interval $[0,\infty)$. Thus we have the following:

Theorem 3.1. *Let* $f(x)\ge 0$ *be a continuous function on* $[0,\infty)$ *satisfying*

$$\int_t^{\infty} f^{\beta}(t)g(t)\mathrm{d}t\ge\int_t^{\infty} t^{\beta}g(t)\mathrm{d}t\qquad\forall\, t\in[0,\infty). \tag{3.1}$$

Then

$$\int_0^\infty f^{\alpha+\beta}(x)g(x)\mathrm{d}x \geq \int_0^\infty x^\beta f^\alpha(x)g(x)\mathrm{d}x \tag{3.2}$$

holds for every positive real number α *and* β*.*

Theorem 3.2. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,\infty)$ *satisfying*

$$\int_t^\infty f(t)g(t)\mathrm{d}t \geq \int_t^\infty tg(t)\mathrm{d}t, \qquad \forall\, t \in [0,\infty). \tag{3.3}$$

Then

$$\int_0^\infty f^{\alpha+\beta}(x)g(x)\mathrm{d}x \geq \int_0^\infty x^\beta f^\alpha(x)g(x)\mathrm{d}x \tag{3.4}$$

holds for every pair of positive real numbers "$\alpha \geq 1$ *and* $\beta > 0$*". Furthermore, for every positive real number "*$\alpha > 0, \beta > 2$*" satisfying (2.15) and (2.16), the inequality (3.4) holds.*

Two more general results follow.

Theorem 3.3. *Let* $f(x) \geq 0$*,* $g(x) \geq 0$ *be two continuous functions on* $[0,\infty)$ *satisfying*

$$\int_t^\infty f^\beta(t)\mathrm{d}t \geq \int_t^\infty g^\beta(t)\mathrm{d}t \qquad \forall\, t \in [0,\infty). \tag{3.5}$$

Furthermore, for any positive real numbers α *and* β*, let* $g(x)$ *be differentiable with* $[g^\alpha(x)]' \geq 0$ *and* $g(0) = 0$*, then*

$$\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x \geq \int_0^\infty g^\beta(x)f^\alpha(x)\mathrm{d}x. \tag{3.6}$$

Proof. Denoting the derivative of $g^\alpha(x)$ by $G(x)$, we obtain,

$$\begin{aligned}\int_0^\infty g^\alpha(x)f^\beta(x)\mathrm{d}x &= \int_0^\infty \left(\int_0^x G(t)\mathrm{d}t\right) f^\beta(x)\mathrm{d}x \\ &= \int_0^\infty G(t)\left(\int_t^\infty f^\beta(x)\mathrm{d}x\right)\mathrm{d}t \\ &\geq \int_0^\infty G(t)\left(\int_t^\infty g^\beta(x)\mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^\infty g^\beta(x)\left(\int_0^x G(t)\mathrm{d}t\right)\mathrm{d}x \\ &= \int_0^\infty g^{\beta+\alpha}(x)\mathrm{d}x.\end{aligned} \tag{3.7}$$

Using Cauchy's inequality, we have

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta}g^{\alpha+\beta}(x) \geq g^\alpha(x)f^\beta(x), \tag{3.8}$$

which, by (3.7), yields

$$\begin{aligned}\frac{\beta}{\alpha+\beta}\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x + \frac{\alpha}{\alpha+\beta}\int_0^\infty g^{\alpha+\beta}(x)\mathrm{d}x &\geq \int_0^\infty g^\alpha(x)f^\beta(x)\mathrm{d}x \\ &\geq \int_0^\infty g^{\beta+\alpha}(x)\mathrm{d}x.\end{aligned} \tag{3.9}$$

The desired result then follows. □

A similar proof yields the following:

Theorem 3.4. *Let* $f(x) \geq 0$, $g(x) \geq 0$ *be two continuous functions on* $[0, \infty)$ *satisfying*

$$\int_t^\infty f(t)\mathrm{d}t \geq \int_t^\infty g(t)\mathrm{d}t, \qquad \forall\, t \in [0, \infty). \tag{3.10}$$

Furthermore, for every pair of positive real numbers satisfying "$\alpha \geq 1$ *and* $\beta > 0$*", let* $g(x)$ *be differentiable with* $[g^\alpha(x)]' \geq 0$ *and* $g(0) = 0$*, then*

$$\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x \geq \int_0^\infty g^\beta(x) f^\alpha(x)\mathrm{d}x. \tag{3.11}$$

Additionally, for every positive real number "$\alpha > 0, \beta > 2$*" satisfying (2.15) and (2.16), inequality (3.11) holds.*

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