FURTHER DEVELOPMENT OF AN OPEN PROBLEM

YU MIAO AND JUNFEN LI
College of Mathematics and Information Science
Henan Normal University
Henan Province, 453007, China
EMail: yumiao728@yahoo.com.cn junfen_li@yahoo.com.cn

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Abstract: In this paper, we generalize an open problem posed by Ngô et al. in [Notes on an Integral Inequality, JIPAM, **7**(4) (2006), Art.120] and give some answers which extend the results of Boukerrioua-Guezane-Lakoud [On an open question regarding an integral inequality, JIPAM, **8**(3) (2007), Art. 77.] and Liu-Li-Dong [On an open problem concerning an integral inequality, JIPAM, **8** (3) (2007), Art. 74.].

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1. Introduction

In [4], the following inequality is found.

Theorem A. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,1]$ *satisfying*

$$\int_x^1 f(t)\mathrm{d}t \geq \int_x^1 t\mathrm{d}t, \qquad \forall\, x \in [0,1], \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)\mathrm{d}x \geq \int_0^1 x^\alpha f(x)\mathrm{d}x, \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)\mathrm{d}x \geq \int_0^1 x f^\alpha(x)\mathrm{d}x, \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

The authors next proposed the following open problem:

Open Problem 1. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,1]$ *satisfying*

$$\int_x^1 f(t)\mathrm{d}t \geq \int_x^1 t\mathrm{d}t, \qquad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x) \geq \int_0^1 x^\alpha f^\beta(x)\mathrm{d}x \tag{1.5}$$

hold for α *and* β*?*

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Several answers and extension results have been given to this open problem [1, 3]. In the present paper, we obtain a generalization of the above Open Problem 1 and provide some resolutions to it. Here, and in what follows, we use X to denote a non-negative random variable (r.v.) on $[0, \infty)$ with probability density function $p(x)$. $\mathbf{E}(X)$ denotes the mathematical expectation of X and $\mathbf{1}_A := \mathbf{1}_A(X)$ denotes the indicator function of the event A. Let $A_t = [t, \infty)$. We now consider the following generalization of Open Problem 1.

Open Problem 2. *Let $f(x) \geq 0$ be a continuous function on $[0, \infty)$. Under what conditions does the inequality*

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^\alpha f^\beta(X)) \tag{1.6}$$

hold for α and β?

Remark 1. Let X possess a uniform distribution on the support interval $[0, 1]$, i.e., the probability density function of X is equal to 1, $x \in [0, 1]$ and zero elsewhere, then the above open problem becomes Open Problem 1.

For convenience, we assume that all the necessary functions in the following are integrable.

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2. Main Results

Theorem 2.1. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ satisfying*

$$\mathbf{E}(f^{\beta}(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X^{\beta}\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.1}$$

Then

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^{\alpha} f^{\beta}(X)) \tag{2.2}$$

holds for every positive real number α and β.

Proof. By Fubini's Theorem and (2.1), we have

$$\begin{aligned}
\mathbf{E}(X^{\alpha} f^{\beta}(X)) &= \int_0^{\infty} x^{\alpha} f^{\beta}(x) p(x)\mathrm{d}x \\
&= \frac{1}{\alpha}\int_0^{\infty}\left(\int_0^{x} t^{\alpha-1}\mathrm{d}t\right) f^{\beta}(x)p(x)\mathrm{d}x \\
&= \frac{1}{\alpha}\int_0^{\infty} t^{\alpha-1}\left(\int_t^{\infty} f^{\beta}(x)p(x)\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{\alpha}\int_0^{\infty} t^{\alpha-1}\mathbf{E}(f^{\beta}(X)\mathbf{1}_{A_t})\mathrm{d}t \\
&\geq \frac{1}{\alpha}\int_0^{\infty} t^{\alpha-1}\mathbf{E}(X^{\beta}\mathbf{1}_{A_t})\mathrm{d}t \\
&= \frac{1}{\alpha}\int_0^{\infty} t^{\alpha-1}\left(\int_t^{\infty} x^{\beta}p(x)\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{\alpha}\int_0^{\infty}\left(\int_0^{x} t^{\alpha-1}\mathrm{d}t\right) x^{\beta}p(x)\mathrm{d}x
\end{aligned} \tag{2.3}$$

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$$= \int_0^\infty x^{\beta+\alpha} p(x) \mathrm{d}x = \mathbf{E}(X^{\beta+\alpha}).$$

Using Cauchy's inequality, we have

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(X) + \frac{\alpha}{\alpha+\beta} X^{\alpha+\beta} \geq X^\alpha f^\beta(X), \tag{2.4}$$

which, by (2.3), yields

$$\frac{\beta}{\alpha+\beta}\mathbf{E}(f^{\alpha+\beta}(X)) + \frac{\alpha}{\alpha+\beta}\mathbf{E}(X^{\alpha+\beta}) \geq \mathbf{E}(X^\alpha f^\beta(X)) \geq \mathbf{E}(X^{\beta+\alpha}). \tag{2.5}$$

□

Remark 2. If we assume that X possesses a uniform distribution on the support interval $[0,1]$ (or $[0,b]$), then the above theorem is Theorem 2.1 (or Theorem 2.4) of Liu-Li-Dong in [3].

Theorem 2.2. *Let $f(x) \geq 0$ be a continuous function on $[0,\infty)$ satisfying*

$$\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.6}$$

Then

$$\mathbf{E}(f^{\alpha+\beta}(X)) \geq \mathbf{E}(X^\beta f^\alpha(X)) \tag{2.7}$$

holds for every positive real number $\alpha \geq 1$ and β.

Proof. By Fubini's Theorem and (2.6), we have

$$\begin{aligned} \mathbf{E}(X^{\alpha+1} f(X)) &= \int_0^\infty x^{\alpha+1} f(x) p(x) \mathrm{d}x \\ &= \frac{1}{\alpha+1}\int_0^\infty \left(\int_0^x t^\alpha \mathrm{d}t\right) f(x)p(x)\mathrm{d}x \end{aligned} \tag{2.8}$$

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$$= \frac{1}{\alpha+1}\int_0^\infty t^\alpha \left(\int_t^\infty f(x)p(x)\mathrm{d}x\right)\mathrm{d}t$$
$$\geq \frac{1}{\alpha+1}\int_0^\infty t^\alpha \left(\int_t^\infty xp(x)\mathrm{d}x\right)\mathrm{d}t$$
$$= \frac{1}{\alpha+1}\int_0^\infty xp(x)\left(\int_0^x t^\alpha \mathrm{d}t\right)\mathrm{d}x = \mathbf{E}(X^{\alpha+2}).$$

Applying Cauchy's inequality, we get

$$\frac{1}{\alpha}f^\alpha(X) + \frac{\alpha-1}{\alpha}X^\alpha \geq f(X)X^{\alpha-1}. \tag{2.9}$$

Multiplying both sides of (2.9) by X^β, we have

$$\frac{1}{\alpha}\mathbf{E}(f^\alpha(X)X^\beta) + \frac{\alpha-1}{\alpha}\mathbf{E}(X^{\alpha+\beta}) \geq \mathbf{E}(f(X)X^{\alpha+\beta-1}) \geq \mathbf{E}(X^{\alpha+\beta}), \tag{2.10}$$

which implies

$$\mathbf{E}(f^\alpha(X)X^\beta) \geq \mathbf{E}(X^{\alpha+\beta}). \tag{2.11}$$

The remainder of the proof is similar to that of Theorem 2.1. □

Remark 3. If we assume that X possesses a uniform distribution on the support interval $[0,1]$ then the above theorem is Theorem 2.3 of Boukerrioua and Guezane-Lakoud in [1].

Next we consider the case "$\alpha > 0, \beta > 2$" by using the ideas of Dragomir-Ngô in [2].

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Lemma 2.3 ([2]). *Let* $f : [a,b] \to [0,\infty)$ *be a continuous function and* $g : [a,b] \to [0,\infty)$ *be non-decreasing, differentiable on* (a,b) *satisfying*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt.$$

Then

$$\int_x^b f^\beta(t)dt \geq \int_x^b g^\beta(t)dt$$

holds for $\beta > 1$.

Lemma 2.4. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,\infty)$ *with* $f'(x) \geq 0$ *on* $(0,\infty)$ *and satisfying*

$$\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.12}$$

Then

$$\mathbf{E}(f^\beta(X)\mathbf{1}_{A_t})) \geq \mathbf{E}(X^\beta \mathbf{1}_{A_t}) \tag{2.13}$$

holds for every positive real number $\beta \geq 1$.

Proof. The proof is a direct extension of Theorem 3 in [2]. □

Theorem 2.5. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,\infty)$ *with* $f'(x) \geq 0$ *on* $(0,\infty)$ *and satisfying*

$$\mathbf{E}(f(X)\mathbf{1}_{A_t}) \geq \mathbf{E}(X\mathbf{1}_{A_t}), \qquad \forall\, t \in [0,\infty). \tag{2.14}$$

In addition, for every positive real number $\alpha > 0, \beta > 2$ *satisfying*

$$\lim_{x\to\infty} f^\alpha(x)x^{\beta-1}\mathbf{E}[(f(X)-X)\mathbf{1}_{A_x}] = 0, \tag{2.15}$$

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and

$$\lim_{x\to\infty} f^{\alpha}(x)x\mathbf{E}[(f^{\beta-1}(X)-X^{\beta-1})\mathbf{1}_{A_x}]=0, \tag{2.16}$$

then

$$\mathbf{E}(f^{\alpha+\beta}(X))\geq \mathbf{E}(X^{\beta}f^{\alpha}(X)). \tag{2.17}$$

Proof. It is obvious that

$$(f(x)-x)(f^{\beta-1}(x)-x^{\beta-1})\geq 0,$$

which implies that

$$f^{\beta+\alpha}(x)\geq x^{\beta-1}f^{1+\alpha}(x)+xf^{\beta-1+\alpha}(x)-x^{\beta}f^{\alpha}(x). \tag{2.18}$$

Integrating by parts and using (2.14) and (2.15), we have

$$\begin{aligned}
&\mathbf{E}(f^{\alpha}(X)X^{\beta-1}(f(X)-X))\\
&=\int_0^{\infty} f^{\alpha}(x)x^{\beta-1}(f(x)-x)p(x)dx\\
&=-\int_0^{\infty} f^{\alpha}(x)x^{\beta-1}\mathrm{d}\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=-f^{\alpha}(x)x^{\beta-1}\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)\Big|_0^{\infty}\\
&\qquad+\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\left(\int_x^{\infty}(f(t)-t)p(t)\mathrm{d}t\right)dx\\
&=\int_0^{\infty}\left(\alpha f'(x)f^{\alpha-1}(x)x^{\beta-1}+(\beta-1)x^{\beta-2}f^{\alpha}(x)\right)\mathbf{E}\left((f(X)-X)\mathbf{1}_{A_x}\right)\mathrm{d}x\geq 0,
\end{aligned}$$

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which yields

$$\mathbf{E}(f^{\alpha+1}(X)X^{\beta-1}) \geq \mathbf{E}(f^{\alpha}(X)X^{\beta}). \tag{2.19}$$

Furthermore, by Lemma 2.4 and the condition (2.16), we have

$$\begin{aligned}
&\mathbf{E}(f^{\alpha}(X)X(f^{\beta-1}(X) - X^{\beta-1})) \\
&= \int_0^\infty f^{\alpha}(x)x(f^{\beta-1}(x) - x^{\beta-1})p(x)dx \\
&= -\int_0^\infty f^{\alpha}(x)x\mathrm{d}\left(\int_x^\infty (f^{\beta-1}(t) - t^{\beta-1})p(t)\mathrm{d}t\right) dx \\
&= -f^{\alpha}(x)x\left(\int_x^\infty (f^{\beta-1}(t) - t^{\beta-1})p(t)\mathrm{d}t\right)\Big|_0^\infty \\
&\qquad + \int_0^\infty (\alpha x f'(x)f^{\alpha-1}(x) + f^{\alpha}(x))\left(\int_x^\infty (f^{\beta-1}(t) - t^{\beta-1})p(t)\mathrm{d}t\right) dx \\
&= \int_0^\infty (\alpha x f'(x)f^{\alpha-1}(x) + f^{\alpha}(x))\mathbf{E}\left((f^{\beta-1}(X) - X^{\beta-1})\mathbf{1}_{A_x}\right) \mathrm{d}x \geq 0,
\end{aligned}$$

which yields

$$\mathbf{E}(f^{\alpha+\beta-1}(X)X) \geq \mathbf{E}(f^{\alpha}(X)X^{\beta}). \tag{2.20}$$

From (2.18)-(2.20), inequality (2.17) holds. □

Further Development of an Open Problem
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3. Further Discussion

Let $g(x) \geq 0$, $0 < \int_0^\infty g(x)\mathrm{d}x < \infty$. If $p(x) := \frac{g(x)}{\int_0^\infty g(x)\mathrm{d}x}$, then it is easy to check that $p(x)$ is a probability density function on the interval $[0, \infty)$. Thus we have the following:

Theorem 3.1. *Let $f(x) \geq 0$ be a continuous function on $[0, \infty)$ satisfying*

$$\int_t^\infty f^\beta(t)g(t)\mathrm{d}t \geq \int_t^\infty t^\beta g(t)\mathrm{d}t \qquad \forall\, t \in [0, \infty). \tag{3.1}$$

Then

$$\int_0^\infty f^{\alpha+\beta}(x)g(x)\mathrm{d}x \geq \int_0^\infty x^\beta f^\alpha(x)g(x)\mathrm{d}x \tag{3.2}$$

holds for every positive real number α and β.

Theorem 3.2. *Let $f(x) \geq 0$ be a continuous function on $[0, \infty)$ satisfying*

$$\int_t^\infty f(t)g(t)\mathrm{d}t \geq \int_t^\infty tg(t)\mathrm{d}t, \qquad \forall\, t \in [0, \infty). \tag{3.3}$$

Then

$$\int_0^\infty f^{\alpha+\beta}(x)g(x)\mathrm{d}x \geq \int_0^\infty x^\beta f^\alpha(x)g(x)\mathrm{d}x \tag{3.4}$$

holds for every pair of positive real numbers "$\alpha \geq 1$ and $\beta > 0$". Furthermore, for every positive real number "$\alpha > 0, \beta > 2$" satisfying (2.15) and (2.16), the inequality (3.4) holds.

Two more general results follow.

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Theorem 3.3. *Let $f(x) \geq 0$, $g(x) \geq 0$ be two continuous functions on $[0,\infty)$ satisfying*

$$\int_t^\infty f^\beta(t)\mathrm{d}t \geq \int_t^\infty g^\beta(t)\mathrm{d}t \qquad \forall\, t \in [0,\infty). \tag{3.5}$$

Furthermore, for any positive real numbers α and β, let $g(x)$ be differentiable with $[g^\alpha(x)]' \geq 0$ and $g(0) = 0$, then

$$\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x \geq \int_0^\infty g^\beta(x) f^\alpha(x)\mathrm{d}x. \tag{3.6}$$

Proof. Denoting the derivative of $g^\alpha(x)$ by $G(x)$, we obtain,

$$\begin{aligned} \int_0^\infty g^\alpha(x) f^\beta(x)\mathrm{d}x &= \int_0^\infty \left(\int_0^x G(t)\mathrm{d}t\right) f^\beta(x)\mathrm{d}x \\ &= \int_0^\infty G(t)\left(\int_t^\infty f^\beta(x)\mathrm{d}x\right)\mathrm{d}t \\ &\geq \int_0^\infty G(t)\left(\int_t^\infty g^\beta(x)\mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^\infty g^\beta(x)\left(\int_0^x G(t)\mathrm{d}t\right)\mathrm{d}x \\ &= \int_0^\infty g^{\beta+\alpha}(x)\mathrm{d}x. \end{aligned} \tag{3.7}$$

Using Cauchy's inequality, we have

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} g^{\alpha+\beta}(x) \geq g^\alpha(x) f^\beta(x), \tag{3.8}$$

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which, by (3.7), yields

$$\frac{\beta}{\alpha+\beta}\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x + \frac{\alpha}{\alpha+\beta}\int_0^\infty g^{\alpha+\beta}(x)\mathrm{d}x \geq \int_0^\infty g^{\alpha}(x) f^{\beta}(x)\mathrm{d}x \geq \int_0^\infty g^{\beta+\alpha}(x)\mathrm{d}x. \tag{3.9}$$

The desired result then follows. □

A similar proof yields the following:

Theorem 3.4. *Let $f(x) \geq 0$, $g(x) \geq 0$ be two continuous functions on $[0,\infty)$ satisfying*

$$\int_t^\infty f(t)\mathrm{d}t \geq \int_t^\infty g(t)\mathrm{d}t, \qquad \forall\, t \in [0,\infty). \tag{3.10}$$

Furthermore, for every pair of positive real numbers satisfying "$\alpha \geq 1$ and $\beta > 0$", let $g(x)$ be differentiable with $[g^{\alpha}(x)]' \geq 0$ and $g(0) = 0$, then

$$\int_0^\infty f^{\alpha+\beta}(x)\mathrm{d}x \geq \int_0^\infty g^{\beta}(x) f^{\alpha}(x)\mathrm{d}x. \tag{3.11}$$

Additionally, for every positive real number "$\alpha > 0, \beta > 2$" satisfying (2.15) and (2.16), inequality (3.11) holds.

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