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APPLICATION LEINDLER SPACES TO THE REAL INTERPOLATION METHOD

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Abstract

The paper is devoted to the important section the Fourier analysis in one variable (AMS subject classification 42A16). In this paper we introduce Leindler space of Fourier - Haar coefficients, so we generalize [2, Theorem 7.a.12] and application to the real method spaces.

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1. Introduction

A Banach space $E[0,1]$ is said to be a *rearrangement invariant* space (r.i) provided $f^*(t) \le g^*(t)$ for any $t \in [0,1]$ and $g \in E$ implies that $f \in E$ and $\|f\|_E \le \|g\|_E$, where $g^*(t)$ is the rearrangement of $|g(t)|$. Denote by φ_E the fundamental function of (r.i) space E such that $\varphi_E = \|\kappa_e(t)\|$ (see, [1, p. 137]). Given $\tau > 0$, the dilation operator $\sigma_\tau f(t) = f(\frac{t}{\tau})$, $t \in [0,1]$ and $\min(1,\tau) \le \|\sigma_\tau\|_{E\to E} \le \max(1,\tau)$. Denote by

$$\alpha_E = \lim_{\tau\to+0} \frac{\ln \|\sigma_\tau\|_{E\to E}}{\ln \tau}, \qquad \beta_E = \lim_{\tau\to\infty} \frac{\ln \|\sigma_\tau\|_{E\to E}}{\ln \tau}$$

the Boyd indices of E. In general, $0 \le \alpha_E \le \beta_E \le 1$.

The associated space to E' is the space of all measurable functions $f(t)$ such that $\int_0^1 f(t)g(t)dt < \infty$ for every $g(t) \in E$ endowed with the norm

$$\|f(t)\|_{E'} = \sup_{\|g(t)\|_E \le 1} \int_0^1 f(t)g(t)dt.$$

For every (r.i) space E space the embedding $E \subset E''$ is isometric. If an (r.i) space E is separable, then (χ_n^k) is everywhere dense in E.

Denote by Ψ the set of increasing concave functions $\psi(t) \ge 0$ on $[0,1]$ with $\psi(0) = 0$. Then each function $\psi(t) \in \Psi$ generates the *Lorentz* space $\Lambda(\psi)$ endowed with the norm

$$\|g(t)\|_{\Lambda(\psi)} = \int_0^1 g^*(t)d\varphi(t) < \infty.$$

For every (r.i) space E space the embedding $E \subset E''$ is isometric.

Let be Ω the set of (n,k) such that $1 \leq k \leq 2^n$, $n \in \mathbf{N} \cup \{0\}$. Put $\chi_0^0 \equiv 1$. If $(n,k) \in \Omega$,

$$\chi_n^k(t) = \begin{cases} 1, & \frac{k-1}{2^n} < t < \frac{2k-1}{2^{n+1}}, \\ -1, & \frac{2k-1}{2^{n+1}} < t < \frac{k}{2^n}, \\ 0, & \text{for any } t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]. \end{cases}$$

The set of functions (χ_n^k) is called the *Haar functions*, normalized in $L_\infty[0,1]$ (see [2, p. 15-18]). If an (r.i) space E is separable, then (χ_n^k) everywhere dense in E. Given $f(t) \in L_1$. The *Fourier-Haar coefficients* are given by

$$c_{n,k}(f) = 2^n \int_0^1 f(t)\chi_n^k(t)dt.$$

Put $g(t) = \sum\limits_{(n,k)\in\Omega} c_{n,k}\chi_n^k$ for any $g \in L_1[0,1]$.

A Banach sequence space E is said to be a *rearrangement invariant* space (r.i) provided that $\|(a_n)\|_E \leq \|(a_n^*)\|_E$,where a_n^* *the rearrangement of sequence* $(a_n)_{n\in\mathbf{N}}$ i.e.

$$a_n^* = \inf\left\{\sup_{i\in\mathbf{N}\setminus\mathbf{J}} |a_i| : \mathbf{J} \subset \mathbf{N}, card(\mathbf{J}) < n\right\}.$$

It is maximal if the unit ball B_E is closed in the poinwise convergence topology inducted by the space A of all real sequences. This condition is equivalent to $E^\# = E'$, where

$$E^\# = \left\{(b_n)_{n\in\mathbf{N}} \subset A : \sum_{n=1}^\infty |a_n b_n| < \infty, (a_n)_{n\in\mathbf{N}} \subset E\right\}$$

is the *Kother dual* of E. Clearly, $E^{\#}$ is a maximal Banach space under the norm

$$\|(b_n)\|_{E^{\#}} = \sup\left\{\sum_{n=1}^{\infty}|a_n b_n| < \infty : \|(a_n)\|_E \le 1\right\}.$$

Denoting $\lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence of positive numbers. We shall use the following notation (see [3, pp. 517-518]):

$$\Lambda_n = \sum_{k=n}^{\infty}\lambda_k \text{ and } \Lambda_n^{(c)} = \sum_{k=n}^{\infty}\lambda_k\Lambda_k^{-c}, (\Lambda_1 < \infty);$$

furthermore, for $c \ge 0$. By analogy with [3, pp. 517-518] we define *Leindler sequence space of Fourier-Haar coefficients*, for $p > 0$, $c \ge 0$, with the norm:

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} = \left(\sum_{n=1}^{\infty}\lambda_n\Lambda_n^{-c}\left(\sum_{k=1}^{2^n}|c_{n,k}|\,2^{-n}\right)^p\right)^{\frac{1}{p}} < \infty.$$

Why do we consider the sequence $(c_{n,k})_{n=1}^{\infty}$? The answer to this question follows from [2, Theorem 7.a.3], i.e. $g \in \Lambda(\psi) \Leftrightarrow \sup_{0<t\le 1} 2^{-\frac{n}{p}}c_{n,1}(g) < \infty$. Here, as usual, $X \hookrightarrow Y$ stands for the continuous embedding, that is, $\|g\|_Y \le C\,\|g\|_X$ for some $C > 0$ and every $g \in X$. The sign $\cong$ means that these spaces coincide to with within equivalence of norms.

2. Problems

By [2, Theorem 7.a.12] for $p = 2$ we have

$$\left\| \sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right\|_{L_2} = \left(\sum_{n=1}^{\infty} 2^{-n} \sum_{k=1}^{2^n} c_{n,k}^2 \right)^{\frac{1}{2}}.$$

If for $\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)}$ we put $p = 2$, $c = 0$, $\lambda_n = 1$, then

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(2,0)} \le M \left\| \sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right\|_{L_2}.$$

Denote by

$$T\left(\sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right) = (c_{n,k})_{(n,k)\in\Omega}.$$

Hence by [1, Chapter 2, §5, Theorem 5.5] we have the operator bounded from $\Lambda(\psi)$ into $\lambda(2,0)$. In general we consider

Problem 1. *Let* $0 < c < 1$, $1 < p < \infty$. *Whether there exists a operator* T *bounded from* $\Lambda(\psi)$ *into* $\lambda(p,c)$*?*

Let (E_0, E_1) be a compatible pair of Banach spaces. We recall

$$K(t,g) = K(t,g,E_0,E_1) = \inf_{g=g_0+g_1,\, g_i\in E_i (i=0,1)} \left(\|g_0\|_{E_0} + t\,\|g_1\|_{E_1} \right).$$

Here $g \in E_0 + E_1$, $0 < t \leq 1$. If $0 < \theta < 1$, $1 \leq p \leq \infty$, then the spaces $(E_0, E_1)_{\theta,p}$ endowed with the norm

$$\|g\|_{(E_0,E_1)_{\theta,p}} = \left(\int_0^1 (K(t,g)t^{-\theta})^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty, \text{ iff } p < \infty$$

and

$$\|g\|_{(E_0,E_1)_{\theta,p}} = \sup_{0<t<1} K(t,g)t^{-\theta} < \infty, \text{ iff } p = \infty$$

are called real method spaces. Let $0 \leq \alpha_0 < \alpha_1 < 1$, $\psi_0(t) = t^{\alpha_0}$, $\psi_{1,}(t) = t^{\alpha_1}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, $\widetilde{\psi}(t) = \frac{t}{\psi(t)}$. In [5, §2, p. 174] the problem was solved: when does the equivalence

$$(\Lambda(\psi_0), \Lambda(\psi_1))_{\theta,p} \cong \left(M(\widetilde{\psi_0}), M(\widetilde{\psi_1})\right)_{\theta,p}.$$

holds?

We consider the embedding $(\Lambda(\psi_0), \Lambda(\psi_1))_{\theta,p} \hookrightarrow \left(M(\widetilde{\psi_0}), M(\widetilde{\psi_1})\right)_{\theta,p}$. Let $0 \leq \alpha_0 = \alpha_1 < 1$, $\psi(t) = t^{\alpha}$, $0 < \theta < 1$, $1 < p \leq \infty$.

Problem 2. *Whether there exists* $0 < c < 1, 1 < p < \infty$ *such that*

$$T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \to (\lambda(p,c), \lambda(p,c))_{\theta,p}\ ?$$

In this article we consider Leindler sequence space of Fourier-Haar coefficients $\lambda(p,c)$.

To prove our theorems we need the following Theorem 1 (see [4]).

Theorem 1. *If $p > 1, 0 \le c < 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} |a_k| \right)^p \le \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_n^{p-c} a_n^p.$$

The constant is best possible.

3. Lemmas and Theorems

Lemma 3.1. *Let* $1 < p < \infty, 0 \le c < 1$ *and* $\sup\limits_{0<t\le 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. *Then the operator* T *is bounded from* $\Lambda(\psi)$ *into* $\lambda(p,c)$.

Proof. By [2, Theorem 4.a.1] for $1 < p < \infty$ we have

$$\int_0^1 \left| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \le \int_0^1 \left| \sum_{n=l}^{\infty} \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \le 2^p \int_0^1 \left| \sum_{(n,k)\in\Omega} c_{n,k} \chi_n^k \right|^p dt,$$

where $n \le l \le \infty$.

On the other hand,

$$\int_0^1 \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right\|_{L_p}^p dt = \int_0^1 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p \, dt = 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p .$$

Therefore,

$$\left(2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p \right)^{\frac{1}{p}} \le 2 \|g\|_{L_p} .$$

From the above and [1, Chapter 2, §5, Theorem 5.5] we get

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} \le 2 \left(\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \right)^{\frac{1}{p}} \|g\|_{\Lambda(\psi)} .$$

Hence the operator T is bounded from $\Lambda(\psi)$ into $\lambda(p,c)$. This proves the assertion. □

Remark 3.1. *In the Lemma 3.1 the condition* $0 < c < 1, 1 < p < \infty$ *is necessary for the operator* T.

We shall formulate the sufficient condition of boundedness of the operator T from $\Lambda(\psi)$ into $\lambda(p, c)$.

Theorem 3.2. *Let* $0 \leq c < 1$, $\sup_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. *For of boundedness the operator* T *bounded from* $\Lambda(\psi)$ *into* $\lambda(p, c)$ *is sufficient that* $2 \leq p < \infty$.

Proof. By Theorem 1 and Hölder's inequality we have

$$\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left(\sum_{(n,k)\in\Omega} |c_{n,k}|^p \, 2^{-n} \right)^{\frac{1}{p}}.$$

Now using [2, Theorem 7.a.12 (c. 2)] and [1, Chapter 2, §5, Theorem 5.5] we obtain that

$$\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left\| \sum_{(n,k)\in\Omega} c_{n,k} \chi_n^k \right\|_{\Lambda(\psi)}.$$

This finishes the proof. □

Remark 3.2. *If* $1 \leq p < 2$, $0 < c < 1$, *then by [2, Theorem 7.a.12 (c. 1)]* $T : \Lambda(\psi) \nrightarrow \lambda(p, c)$.

Theorem 3.3. *Let* $0 \leq c < 1, 2 \leq p \leq \infty$, $\sup_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. *Then*

$$T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \to (\lambda(p, c), \lambda(p, c))_{\theta,p}.$$

Proof. Clearly, by Hölder's inequality the estimate

$$\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)} \leq \frac{p}{1-c}\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{p}-1}\Lambda_n^{1-\frac{c}{p}}\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\ell_2}$$

holds. It is known that the operator T is bounded from L_2 into ℓ_2. Then from the above and [1, Chapter 2, §5, Theorem 5.5] we obtain

$$K(t,(c_{n,k})_{n=1}^{\infty},\lambda(p,c),\lambda(p,c)) \leq K(t,g,\Lambda(\psi),\Lambda(\psi)).$$

Hence $T:(\Lambda(\psi),\Lambda(\psi))_{\theta,p} \to (\lambda(p,c),\lambda(p,c))_{\theta,p}$. This completes the proof. □

References

[1] S.G. KREIN, YU. I. PETUNIN AND E.M. SEMENOV, *Interpolation linear operators*, Nauka, Moscow, 1978, in Russian; Math. Mono., Amer. Math. Soc., Providence, RI, 1982, English translation.

[2] I. NOVIKOV AND E. SEMENOV, *Haar Series and Linear Operators*, Mathematics and Its Applications, Kluwer Acad. Publ., 1997.

[3] L. LEINDLER, Hardy - Bennett - Type Theorems, *Math. Ineq. and Appl.*, **4** (1998), 517–526.

[4] L. LEINDLER, Two theorems of Hardy-Bennett - type, *Acta Math. Hung.*, **79**(4) (1998), 341–350.

[5] E. SEMENOV, On the stability of the real interpolation method in the class of rearrangement invariant spaces, *Israel Math. Proceedings*, **13** (1999), 172–182.