

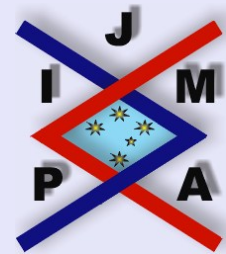
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APPLICATION LEINDLER SPACES TO THE REAL INTERPOLATION METHOD

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

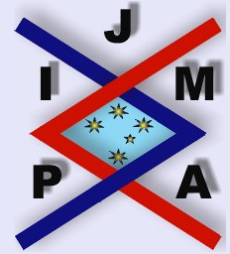
The paper is devoted to the important section the Fourier analysis in one variable (AMS subject classification 42A16). In this paper we introduce Leindler space of Fourier - Haar coefficients, so we generalize [2, Theorem 7.a.12] and application to the real method spaces.

2000 Mathematics Subject Classification: 26D15, 40A05, 42A16, 40A99, 46E30, 47A30, 47A63.

Key words: Leindler sequence space of Fourier - Haar coefficients, Lorentz space, Haar functions, real method spaces.

Contents

1	Introduction	3
2	Problems	6
3	Lemmas and Theorems	9
	References	



Application Leindler Spaces to the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 2 of 12

1. Introduction

A Banach space $E[0, 1]$ is said to be a *rearrangement invariant* space (r.i) provided $f^*(t) \leq g^*(t)$ for any $t \in [0, 1]$ and $g \in E$ implies that $f \in E$ and $\|f\|_E \leq \|g\|_E$, where $g^*(t)$ is the rearrangement of $|g(t)|$. Denote by φ_E the fundamental function of (r.i) space E such that $\varphi_E = \|\kappa_e(t)\|$ (see, [1, p. 137]). Given $\tau > 0$, the dilation operator $\sigma_\tau f(t) = f(\frac{t}{\tau})$, $t \in [0, 1]$ and $\min(1, \tau) \leq \|\sigma_\tau\|_{E \rightarrow E} \leq \max(1, \tau)$. Denote by

$$\alpha_E = \lim_{\tau \rightarrow +0} \frac{\ln \|\sigma_\tau\|_{E \rightarrow E}}{\ln \tau}, \quad \beta_E = \lim_{\tau \rightarrow \infty} \frac{\ln \|\sigma_\tau\|_{E \rightarrow E}}{\ln \tau}$$

the Boyd indices of E . In general, $0 \leq \alpha_E \leq \beta_E \leq 1$.

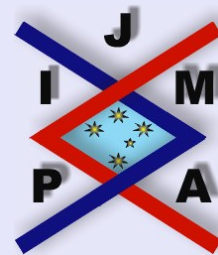
The associated space to E' is the space of all measurable functions $f(t)$ such that $\int_0^1 f(t)g(t)dt < \infty$ for every $g(t) \in E$ endowed with the norm

$$\|f(t)\|_{E'} = \sup_{\|g(t)\|_E \leq 1} \int_0^1 f(t)g(t)dt.$$

For every (r.i) space E space the embedding $E \subset E''$ is isometric. If an (r.i) space E is separable, then (χ_n^k) is everywhere dense in E .

Denote by Ψ the set of increasing concave functions $\psi(t) \geq 0$ on $[0, 1]$ with $\psi(0) = 0$. Then each function $\psi(t) \in \Psi$ generates the *Lorentz* space $\Lambda(\psi)$ endowed with the norm

$$\|g(t)\|_{\Lambda(\psi)} = \int_0^1 g^*(t)d\varphi(t) < \infty.$$



For every (r.i) space E space the embedding $E \subset E''$ is isometric.

Let be Ω the set of (n, k) such that $1 \leq k \leq 2^n, n \in \mathbf{N} \cup \{0\}$. Put $\chi_0^0 \equiv 1$. If $(n, k) \in \Omega$,

$$\chi_n^k(t) = \begin{cases} 1, & \frac{k-1}{2^n} < t < \frac{2k-1}{2^{n+1}}, \\ -1, & \frac{2k-1}{2^{n+1}} < t < \frac{k}{2^n}, \\ 0, & \text{for any } t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]. \end{cases}$$

The set of functions (χ_n^k) is called the *Haar functions*, normalized in $L_\infty[0, 1]$ (see [2, p. 15-18]). If an (r.i) space E is separable, then (χ_n^k) everywhere dense in E . Given $f(t) \in L_1$. The *Fourier-Haar coefficients* are given by

$$c_{n,k}(f) = 2^n \int_0^1 f(t) \chi_n^k(t) dt.$$

Put $g(t) = \sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k$ for any $g \in L_1[0, 1]$.

A Banach sequence space E is said to be a *rearrangement invariant space* (r.i) provided that $\|(a_n)\|_E \leq \|(a_n^*)\|_E$, where a_n^* the *rearrangement of sequence* $(a_n)_{n \in \mathbf{N}}$ i.e.

$$a_n^* = \inf \left\{ \sup_{i \in \mathbf{N} \setminus \mathbf{J}} |a_i| : \mathbf{J} \subset \mathbf{N}, \text{card}(\mathbf{J}) < n \right\}.$$

It is maximal if the unit ball B_E is closed in the poinwise convergence topology induced by the space A of all real sequences. This condition is equivalent to $E^\# = E'$, where

$$E^\# = \left\{ (b_n)_{n \in \mathbf{N}} \subset A : \sum_{n=1}^{\infty} |a_n b_n| < \infty, (a_n)_{n \in \mathbf{N}} \subset E \right\}$$



Application Leindler Spaces to the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 4 of 12

is the *Kother dual* of E . Clearly, $E^\#$ is a maximal Banach space under the norm

$$\|(b_n)\|_{E^\#} = \sup \left\{ \sum_{n=1}^{\infty} |a_n b_n| < \infty : \|(a_n)\|_E \leq 1 \right\}.$$

Denoting $\lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence of positive numbers. We shall use the following notation (see [3, pp. 517-518]):

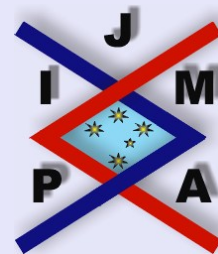
$$\Lambda_n = \sum_{k=n}^{\infty} \lambda_k \text{ and } \Lambda_n^{(c)} = \sum_{k=n}^{\infty} \lambda_k \Lambda_k^{-c}, (\Lambda_1 < \infty);$$

furthermore, for $c \geq 0$. By analogy with [3, pp. 517-518] we define *Leindler sequence space of Fourier-Haar coefficients*, for $p > 0, c \geq 0$, with the norm:

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} = \left(\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{2^n} |c_{n,k}| 2^{-n} \right)^p \right)^{\frac{1}{p}} < \infty.$$

Why do we consider the sequence $(c_{n,k})_{n=1}^{\infty}$? The answer to this question follows from [2, Theorem 7.a.3], i.e. $g \in \Lambda(\psi) \Leftrightarrow \sup_{0 < t \leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$.

Here, as usual, $X \hookrightarrow Y$ stands for the continuous embedding, that is, $\|g\|_Y \leq C \|g\|_X$ for some $C > 0$ and every $g \in X$. The sign \cong means that these spaces coincide to within equivalence of norms.



Application Leindler Spaces to the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 5 of 12

2. Problems

By [2, Theorem 7.a.12] for $p = 2$ we have

$$\left\| \sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k \right\|_{L_2} = \left(\sum_{n=1}^{\infty} 2^{-n} \sum_{k=1}^{2^n} c_{n,k}^2 \right)^{\frac{1}{2}}.$$

If for $\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)}$ we put $p = 2, c = 0, \lambda_n = 1$, then

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(2,0)} \leq M \left\| \sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k \right\|_{L_2}.$$

Denote by

$$T \left(\sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k \right) = (c_{n,k})_{(n,k) \in \Omega}.$$

Hence by [1, Chapter 2, §5, Theorem 5.5] we have the operator bounded from $\Lambda(\psi)$ into $\lambda(2, 0)$. In general we consider

Problem 1. *Let $0 < c < 1, 1 < p < \infty$. Whether there exists a operator T bounded from $\Lambda(\psi)$ into $\lambda(p, c)$?*

Let (E_0, E_1) be a compatible pair of Banach spaces. We recall

$$K(t, g) = K(t, g, E_0, E_1) = \inf_{g=g_0+g_1, g_i \in E_i (i=0,1)} (\|g_0\|_{E_0} + t \|g_1\|_{E_1}).$$



Application Leindler Spaces to
the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 6 of 12

Here $g \in E_0 + E_1$, $0 < t \leq 1$. If $0 < \theta < 1$, $1 \leq p \leq \infty$, then the spaces $(E_0, E_1)_{\theta, p}$ endowed with the norm

$$\|g\|_{(E_0, E_1)_{\theta, p}} = \left(\int_0^1 (K(t, g)t^{-\theta})^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty, \text{ iff } p < \infty$$

and

$$\|g\|_{(E_0, E_1)_{\theta, p}} = \sup_{0 < t < 1} K(t, g)t^{-\theta} < \infty, \text{ iff } p = \infty$$

are called real method spaces. Let $0 \leq \alpha_0 < \alpha_1 < 1$, $\psi_0(t) = t^{\alpha_0}$, $\psi_1(t) = t^{\alpha_1}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, $\tilde{\psi}(t) = \frac{t}{\psi(t)}$. In [5, §2, p. 174] the problem was solved: when does the equivalence

$$(\Lambda(\psi_0), \Lambda(\psi_1))_{\theta, p} \cong (M(\tilde{\psi}_0), M(\tilde{\psi}_1))_{\theta, p} .$$

holds?

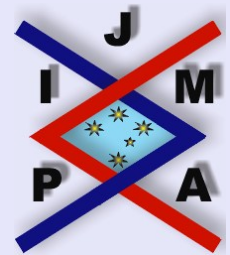
We consider the embedding $(\Lambda(\psi_0), \Lambda(\psi_1))_{\theta, p} \hookrightarrow (M(\tilde{\psi}_0), M(\tilde{\psi}_1))_{\theta, p}$. Let $0 \leq \alpha_0 = \alpha_1 < 1$, $\psi(t) = t^\alpha$, $0 < \theta < 1$, $1 < p \leq \infty$.

Problem 2. *Whether there exists $0 < c < 1$, $1 < p < \infty$ such that*

$$T : (\Lambda(\psi), \Lambda(\psi))_{\theta, p} \rightarrow (\lambda(p, c), \lambda(p, c))_{\theta, p} ?$$

In this article we consider Leindler sequence space of Fourier-Haar coefficients $\lambda(p, c)$.

To prove our theorems we need the following Theorem 1 (see [4]).



Application Leindler Spaces to the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

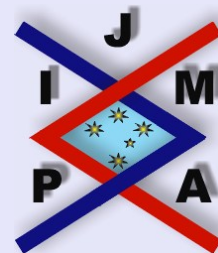
Quit

Page 7 of 12

Theorem 1. *If $p > 1, 0 \leq c < 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^n |a_k| \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_n^{p-c} a_n^p.$$

The constant is best possible.



**Application Leindler Spaces to
the Real Interpolation Method**

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 8 of 12

3. Lemmas and Theorems

Lemma 3.1. Let $1 < p < \infty$, $0 \leq c < 1$ and $\sup_{0 < t \leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. Then the operator T is bounded from $\Lambda(\psi)$ into $\lambda(p, c)$.

Proof. By [2, Theorem 4.a.1] for $1 < p < \infty$ we have

$$\int_0^1 \left| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \leq \int_0^1 \left| \sum_{n=l}^{\infty} \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \leq 2^p \int_0^1 \left| \sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k \right|^p dt,$$

where $n \leq l \leq \infty$.

On the other hand,

$$\int_0^1 \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right\|_{L_p}^p dt = \int_0^1 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p dt = 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p.$$

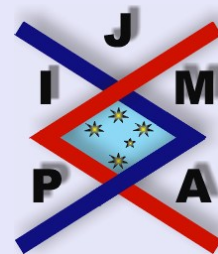
Therefore,

$$\left(2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p \right)^{\frac{1}{p}} \leq 2 \|g\|_{L_p}.$$

From the above and [1, Chapter 2, §5, Theorem 5.5] we get

$$\| (c_{n,k})_{n=1}^{\infty} \|_{\lambda(p,c)} \leq 2 \left(\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \right)^{\frac{1}{p}} \|g\|_{\Lambda(\psi)}.$$

Hence the operator T is bounded from $\Lambda(\psi)$ into $\lambda(p, c)$. This proves the assertion. \square



Application Leindler Spaces to
the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 9 of 12

Remark 3.1. In the Lemma 3.1 the condition $0 < c < 1, 1 < p < \infty$ is necessary for the operator T .

We shall formulate the sufficient condition of boundedness of the operator T from $\Lambda(\psi)$ into $\lambda(p, c)$.

Theorem 3.2. Let $0 \leq c < 1, \sup_{0 < t \leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. For of boundedness the operator T bounded from $\Lambda(\psi)$ into $\lambda(p, c)$ is sufficient that $2 \leq p < \infty$.

Proof. By Theorem 1 and Hölder's inequality we have

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left(\sum_{(n,k) \in \Omega} |c_{n,k}|^p 2^{-n} \right)^{\frac{1}{p}}.$$

Now using [2, Theorem 7.a.12 (c. 2)] and [1, Chapter 2, §5, Theorem 5.5] we obtain that

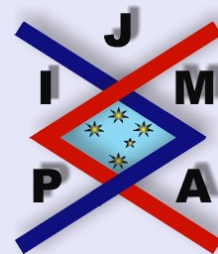
$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left\| \sum_{(n,k) \in \Omega} c_{n,k} \chi_n^k \right\|_{\Lambda(\psi)}.$$

This finishes the proof. □

Remark 3.2. If $1 \leq p < 2, 0 < c < 1$, then by [2, Theorem 7.a.12 (c. 1)] $T : \Lambda(\psi) \not\rightarrow \lambda(p, c)$.

Theorem 3.3. Let $0 \leq c < 1, 2 \leq p \leq \infty, \sup_{0 < t \leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. Then

$$T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \rightarrow (\lambda(p, c), \lambda(p, c))_{\theta,p}.$$



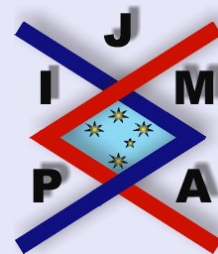
Proof. Clearly, by Hölder's inequality the estimate

$$\| (c_{n,k})_{n=1}^{\infty} \|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \| (c_{n,k})_{n=1}^{\infty} \|_{\ell_2}$$

holds. It is known that the operator T is bounded from L_2 into ℓ_2 . Then from the above and [1, Chapter 2, §5, Theorem 5.5] we obtain

$$K(t, (c_{n,k})_{n=1}^{\infty}, \lambda(p, c), \lambda(p, c)) \leq K(t, g, \Lambda(\psi), \Lambda(\psi)).$$

Hence $T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \rightarrow (\lambda(p, c), \lambda(p, c))_{\theta,p}$. This completes the proof. \square



**Application Leindler Spaces to
the Real Interpolation Method**

Vadim Kuklin

Title Page

Contents



Go Back

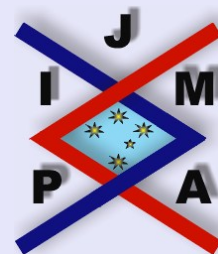
Close

Quit

Page 11 of 12

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Application Leindler Spaces to
the Real Interpolation Method

Vadim Kuklin

Title Page

Contents



Go Back

Close

Quit

Page 12 of 12