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INTEGRAL MEANS INEQUALITIES FOR FRACTIONAL DERIVATIVES OF SOME GENERAL SUBCLASSES OF ANALYTIC FUNCTIONS

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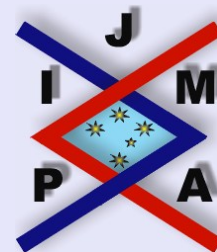
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Abstract

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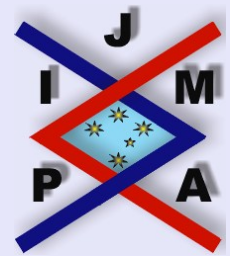


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Abstract

Integral means inequalities are obtained for the fractional derivatives of order $p + \lambda$ ($0 \leq p \leq n$; $0 \leq \lambda < 1$) of functions belonging to certain general subclasses of analytic functions. Relevant connections with various known integral means inequalities are also pointed out.

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1. Introduction, Definitions, and Preliminaries

Let \mathcal{A} denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

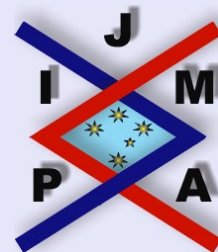
Also let $\mathcal{A}(n)$ denote the subclass of \mathcal{A} consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0; \quad n \in \mathbb{N} := \{1, 2, 3, \dots\}).$$

We denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ of functions which are *univalent* in \mathbb{U} , and by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the subclasses of $\mathcal{T}(n)$ consisting of functions which are, respectively, *starlike of order* α ($0 \leq \alpha < 1$) and *convex of order* α ($0 \leq \alpha < 1$) in \mathbb{U} . The classes $\mathcal{A}(n)$, $\mathcal{T}(n)$, $\mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$ were investigated by Chatterjea [1] (and Srivastava *et al.* [9]). In particular, the following subclasses:

$$\mathcal{T} := \mathcal{T}(1), \quad \mathcal{T}^*(\alpha) := \mathcal{T}_\alpha(1), \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{C}_\alpha(1)$$

were considered earlier by Silverman [7].



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Next, following the work of Sekine and Owa [4], we denote by $\mathcal{A}(n, \vartheta)$ the subclass of \mathcal{A} consisting of all functions $f(z)$ of the form:

$$(1.1) \quad f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^k \quad (\vartheta \in \mathbb{R}; a_k \geq 0; n \in \mathbb{N}).$$

Finally, the subclasses $\mathcal{T}(n, \vartheta)$, $\mathcal{T}_{\alpha}^*(n, \vartheta)$, and $\mathcal{C}_{\alpha}(n, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ are defined in the same way as the subclasses $\mathcal{T}(n)$, $\mathcal{T}_{\alpha}(n)$, and $\mathcal{C}_{\alpha}(n)$ of the class $\mathcal{A}(n)$.

We begin by recalling the following useful characterizations of the function classes $\mathcal{T}_{\alpha}^*(n, \vartheta)$ and $\mathcal{C}_{\alpha}(n, \vartheta)$ (see Sekine and Owa [4]).

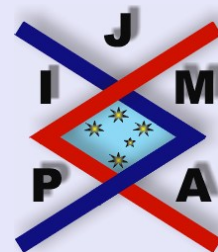
Lemma 1.1. *A function $f(z) \in \mathcal{A}(n, \vartheta)$ of the form (1.1) is in the class $\mathcal{T}_{\alpha}^*(n, \vartheta)$ if and only if*

$$(1.2) \quad \sum_{k=n+1}^{\infty} (k - \alpha) a_k \leq 1 - \alpha \quad (n \in \mathbb{N}; 0 \leq \alpha < 1).$$

Lemma 1.2. *A function $f(z) \in \mathcal{A}(n, \vartheta)$ of the form (1.1) is in the class $\mathcal{C}_{\alpha}(n, \vartheta)$ if and only if*

$$(1.3) \quad \sum_{k=n+1}^{\infty} k(k - \alpha) a_k \leq 1 - \alpha \quad (n \in \mathbb{N}; 0 \leq \alpha < 1).$$

Motivated by the equalities in (1.2) and (1.3) above, Sekine *et al.* [6] defined a general subclass $\mathcal{A}(n; B_k, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ consisting of functions $f(z)$



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of the form (1.1), which satisfy the following inequality:

$$\sum_{k=n+1}^{\infty} B_k a_k \leq 1 \quad (B_k > 0; n \in \mathbb{N}).$$

Thus it is easy to verify each of the following classifications and relationships:

$$\mathcal{A}(n; k, \vartheta) = \mathcal{T}_0^*(n, \vartheta) =: \mathcal{T}^*(n, \vartheta) = \mathcal{T}(n, \vartheta),$$

$$\mathcal{A}\left(n; \frac{k - \alpha}{1 - \alpha}, \vartheta\right) = \mathcal{T}_\alpha^*(n, \vartheta) \quad (0 \leq \alpha < 1),$$

and

$$\mathcal{A}\left(n; \frac{k(k - \alpha)}{1 - \alpha}, \vartheta\right) = \mathcal{C}_\alpha(n, \vartheta) \quad (0 \leq \alpha < 1).$$

As a matter of fact, Sekine *et al.* [6] also obtained each of the following basic properties of the general classes $\mathcal{A}(n; B_k, \vartheta)$.

Theorem 1.3. $\mathcal{A}(n; B_k, \vartheta)$ is the convex subfamily of the class $\mathcal{A}(n, \vartheta)$.

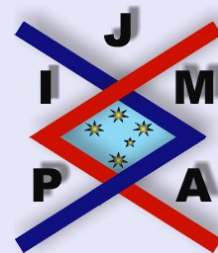
Theorem 1.4. Let

$$(1.4) \quad f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{e^{i(k-1)\vartheta}}{B_k} z^k$$

$$(k = n + 1, n + 2, n + 3, \dots; n \in \mathbb{N}).$$

Then $f \in \mathcal{A}(n; B_k, \vartheta)$ if and only if $f(z)$ can be expressed as follows:

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z),$$



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where

$$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1 \quad (\lambda_1 \geq 0; \lambda_k \geq 0; n \in \mathbb{N}).$$

Corollary 1.5. *The extreme points of the class $\mathcal{A}(n; B_k, \vartheta)$ are the functions $f_1(z)$ and $f_k(z)$ ($k \geq n + 1; n \in \mathbb{N}$) given by (1.4).*

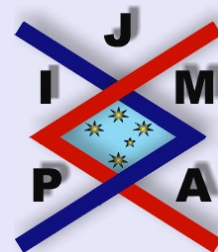
Applying the concepts of extreme points, fractional calculus, and subordination, Sekine *et al.* [6] obtained several integral means inequalities for higher-order fractional derivatives and fractional integrals of functions belonging to the general classes $\mathcal{A}(n; B_k, \vartheta)$. Subsequently, Sekine and Owa [5] discussed the weakening of the hypotheses for B_k in those results by Sekine *et al.* [6]. In this paper, we investigate the integral means inequalities for the fractional derivatives of $f(z)$ of a general order $p + \lambda$ ($0 \leq p \leq n; 0 \leq \lambda < 1$) of functions $f(z)$ belonging to the general classes $\mathcal{A}(n; B_k, \vartheta)$.

We shall make use of the following definitions of fractional derivatives (*cf.* Owa [3]; see also Srivastava and Owa [8]).

Definition 1.1. *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$(1.5) \quad D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin and the multiplicity of $(z - \zeta)^{-\lambda}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.



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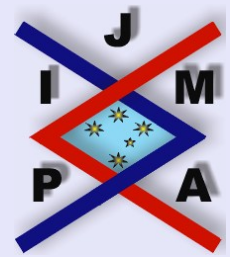


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Definition 1.2. Under the hypotheses of Definition 1.1, the fractional derivative of order $n + \lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^\lambda f(z) \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

It readily follows from (1.5) in Definition 1.1 that

$$(1.6) \quad D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leq \lambda < 1).$$

We shall also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in \mathbb{U} , the function $f(z)$ is said to be *subordinate* to $g(z)$ in \mathbb{U} if there exists a function $w(z)$, analytic in \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Theorem 1.6 (Littlewood [2]). If the functions $f(z)$ and $g(z)$ are analytic in \mathbb{U} with

$$g(z) \prec f(z),$$

then

$$\int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \quad (\mu > 0; 0 < r < 1).$$

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2. The Main Integral Means Inequalities

Theorem 2.1. Suppose that $f(z) \in \mathcal{A}(n; k^{p+1}B_k, \vartheta)$ and that

$$\frac{(h+1)^q B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \leq B_k \quad (k \geq n+1)$$

for some $h \geq n$, $0 \leq \lambda < 1$, and $0 \leq q \leq p \leq n$. Also let the function $f_{h+1}(z)$ be defined by

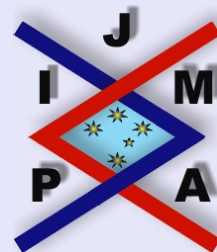
$$(2.1) \quad f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} z^{h+1} \quad (f_{h+1} \in A(n; k^{q+1}B_k, \vartheta)).$$

Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$(2.2) \quad \int_0^{2\pi} |D_z^{p+\lambda} f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^{p+\lambda} f_{h+1}(z)|^\mu d\theta \quad (0 \leq \lambda < 1; \mu > 0).$$

Proof. By virtue of the fractional derivative formula (1.6) and Definition 1.2, we find from (1.1) that

$$\begin{aligned} & D_z^{p+\lambda} f(z) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} a_k z^{k-1} \right) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1} \right), \end{aligned}$$



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where

$$(2.3) \quad \Phi(k) := \frac{\Gamma(k-p)}{\Gamma(k+1-\lambda-p)} \quad (0 \leq \lambda < 1; k \geq n+1; n \in \mathbb{N}).$$

Since $\Phi(k)$ is a *decreasing* function of k , we have

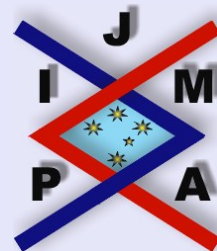
$$0 < \Phi(k) \leq \Phi(n+1) = \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \\ (0 \leq \lambda < 1; k \geq n+1; n \in \mathbb{N}).$$

Similarly, from (2.1), (1.6), and Definition 1.2, we obtain, for $0 \leq \lambda < 1$,

$$D_z^{p+\lambda} f_{h+1}(z) \\ = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h \right).$$

For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1} \right|^\mu d\theta \\ \leq \int_0^{2\pi} \left| 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h \right|^\mu d\theta, \\ (0 \leq \lambda < 1; \mu > 0).$$



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Thus, by applying Theorem 1.6, it would suffice to show that

$$(2.4) \quad 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2 - \lambda - p) \frac{k!}{(k - p - 1)!} \Phi(k) a_k z^{k-1} \\ < 1 - \frac{e^{ih\vartheta}}{(h + 1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2 - \lambda - p) \Gamma(h + 2)}{\Gamma(h + 2 - \lambda - p)} z^h.$$

Indeed, by setting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2 - \lambda - p) \frac{k!}{(k - p - 1)!} \Phi(k) a_k z^{k-1} \\ = 1 - \frac{e^{ih\vartheta}}{(h + 1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2 - \lambda - p) \Gamma(h + 2)}{\Gamma(h + 2 - \lambda - p)} \{w(z)\}^h,$$

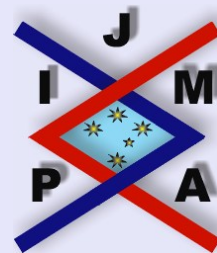
we find that

$$\{w(z)\}^h = \frac{(h + 1)^{q+1} B_{h+1} \Gamma(h + 2 - \lambda - p)}{e^{ih\vartheta} \Gamma(h + 2)} \\ \times \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \frac{k!}{(k - p - 1)!} \Phi(k) a_k z^{k-1},$$

which readily yields $w(0) = 0$.

Therefore, we have

$$|w(z)|^h \leq \frac{(h + 1)^{q+1} B_{h+1} \Gamma(h + 2 - \lambda - p)}{\Gamma(h + 2)} \sum_{k=n+1}^{\infty} \frac{k!}{(k - p - 1)!} \Phi(k) a_k |z|^{k-1}$$



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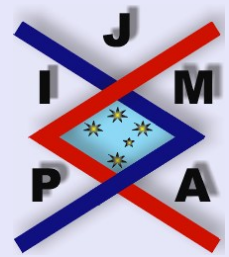


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$$\begin{aligned}
 &\leq |z|^n \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \\
 &\quad \times \Phi(n+1) \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k \\
 &= |z|^n \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \\
 &\quad \times \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k \\
 &= |z|^n \frac{(h+1)^q B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \\
 &\quad \times \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k \\
 &\leq |z|^n \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} B_k a_k \\
 (2.5) \quad &\leq |z|^n \sum_{k=n+1}^{\infty} k^{p+1} B_k a_k \leq |z|^n < 1 \quad (n \in \mathbb{N}),
 \end{aligned}$$

by means of the hypothesis of Theorem 2.1.

In light of the last inequality in (2.5) above, we have the subordination (2.4), which evidently proves Theorem 2.1. \square

3. Remarks and Observations

First of all, in its special case when $p = q = 0$, Theorem 2.1 readily yields

Corollary 3.1 (cf. Sekine and Owa [5], Theorem 6). *Suppose that $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ and that*

$$\frac{B_{h+1}\Gamma(h+2-\lambda)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+2-\lambda)} \leq B_k \quad (k \geq n+1; n \in \mathbb{N})$$

for some $h \geq n$ and $0 \leq \lambda < 1$. Also let the function $f_{h+1}(z)$ be defined by

$$(3.1) \quad f_{h+1}(z) = z - \frac{e^{i h \vartheta}}{(h+1)B_{h+1}} z^{h+1} \quad (f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)).$$

Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$(3.2) \quad \int_0^{2\pi} |D_z^\lambda f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^\lambda f_{h+1}(z)|^\mu d\theta \quad (0 \leq \lambda < 1; \mu > 0).$$

A further consequence of Corollary 3.1 when $h = n$ would lead us immediately to Corollary 3.2 below.

Corollary 3.2. *Suppose that $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ and that*

$$(3.3) \quad B_{n+1} \leq B_k \quad (k \geq n+1; n \in \mathbb{N}).$$

Also let the function $f_{n+1}(z)$ be defined by

$$f_{n+1}(z) = z - \frac{e^{in\vartheta}}{(n+1)B_{n+1}} z^{n+1} \quad (f_{n+1} \in \mathcal{A}(n; kB_k, \vartheta)).$$



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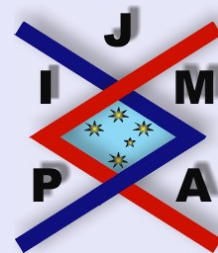
Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} |D_z^\lambda f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^\lambda f_{n+1}(z)|^\mu d\theta \quad (0 \leq \lambda < 1; \mu > 0).$$

The hypothesis (3.3) in Corollary 3.2 is weaker than the corresponding hypothesis in an earlier result of Sekine *et al.* [6, p. 953, Theorem 6].

Next, for $p = 1$ and $q = 0$, Theorem 2.1 reduces to an integral means inequality of Sekine and Owa [5, Theorem 7] which, for $h = n$, yields another result of Sekine *et al.* [6, p. 953, Theorem 7] under weaker hypothesis as mentioned above.

Finally, by setting $p = q = 1$ in Theorem 2.1, we obtain a slightly improved version of another integral means inequalities of Sekine and Owa [5, Theorem 8] with respect to the parameter λ (see also Sekine *et al.* [6, p. 955, Theorem 8] for the case when $h = n$, just as we remarked above).



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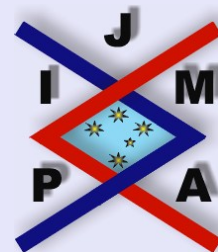
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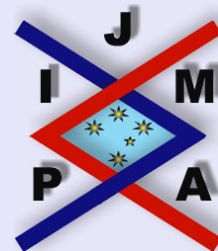
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