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HYERS-ULAM STABILITY OF THE GENERALIZED QUADRATIC FUNCTIONAL EQUATION IN AMENABLE SEMIGROUPS

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ABSTRACT. In this paper we derive the Hyers-Ulam stability of the quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G,$$

respectively the functional equation

$$f(xy) + g(x\sigma(y)) = f(x) + g(y), \quad x, y \in G,$$

where G is an amenable semigroup, σ is a morphism of G such that $\sigma \circ \sigma = I$, respectively where G is an amenable semigroup and σ is an homomorphism of G such that $\sigma \circ \sigma = I$.

Key words and phrases: Hyers-Ulam stability, Quadratic functional equation, Amenable semigroup, Morphism of semigroup.

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1. INTRODUCTION

In 1940, Ulam [21] raised a question concerning the stability problem of group homomorphisms:

> Given a group G_1, a metric group (G_2, d), a number $\varepsilon > 0$ and a mapping $f : G_1 \longrightarrow G_2$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, does there exist a homomorphism $h : G_1 \longrightarrow G_2$ and a constant

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$k > 0$, depending only on G_1 and G_2 such that $d(f(x), h(x)) \leq k\varepsilon$ for all x in G_1?

The case of approximately additive mappings was solved by D. H. Hyers [8] under the assumption that G_1 and G_2 are Banach spaces.

In 1978, Th. M. Rassias [16] gave a remarkable generalization of the Hyers's result which allows the Cauchy difference to be unbounded. Since then, several mathematicians have been attracted to the results of Hyers and Rassias and investigated a number of stability problems of different functional equations. See for example the monographs of the following references [5, 6, 9, 10, 11, 16].

The quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in G \tag{1.1}$$

has been much studied. It was generalized by Stetkær [19] to the more general equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G, \tag{1.2}$$

where σ is an automorphism of the abelian group G such that $\sigma^2 = I$, (I denotes the identity).

A stability result for the quadratic functional equation (1.1) was derived by Skof [18], Cholewa [3] and by Czerwik [4].

Recently, Bouikhalene, Elqorachi and Rassias stated the stability theorem of equation (1.2), see [1] and [2].

Székelyhidi [20] extended the Hyers's result to amenable semigroups. He replaced the original proof given by Hyers by a new one based on the use of invariant means.

In [22] Yang obtained the stability of the quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad x, y \in G, \tag{1.3}$$

in amenable groups.

The purpose of the present paper is a joint treatment of the functional equations (1.1) and (1.3) and their generalization, where the unifying object is a morphism like the one introduced in (1.2).

New features of the paper:

(1) In comparison with [20] and [22], we work with a general morphism σ.
(2) In contrast to [1] and [2] we here allow the (semi)group G to be non-abelian.

In Section 2, we obtain the stability of the quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G, \tag{1.4}$$

where σ is a morphism of G such that $\sigma \circ \sigma = I$. The result of this section can be compared with the ones of Yang [22] because we formulate them in the same way by using some ideas from [22].

In Section 3, we obtain the stability of the generalized quadratic functional equation

$$f(xy) + g(x\sigma(y)) = f(x) + g(y), \quad x, y \in G, \tag{1.5}$$

where σ is an automorphism of G such that $\sigma \circ \sigma = I$.

2. STABILITY OF EQUATION (1.4) IN AMENABLE SEMIGROUPS

In this section we investigate the Hyers-Ulam stability of the quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G, \tag{2.1}$$

where G is an amenable semigroup with unit element e and $\sigma : G \longrightarrow G$ is a morphism of G, i.e. σ is an antiautomorphism: $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$ or σ is an automorphism:

$\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in G$. Furthermore, we assume that σ satisfies $(\sigma \circ \sigma)(x) = x$, for all $x \in G$.

We recall that a semigroup G is said to be amenable if there exists an invariant mean on the space of the bounded complex functions defined on G. We refer to [7] for the definition and properties of invariant means.

Throughout this paper, as in [5], we use the following definition.

Definition 2.1. Let G be a semigroup and B a Banach space. We say that the equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G \tag{2.2}$$

is stable for the pair (G, B) if for every function $f : G \longrightarrow B$ such that

$$\left\| \frac{1}{2}[f(xy) + f(x\sigma(y))] - f(x) - f(y) \right\| \leq \delta, \quad x, y \in G \text{ for some } \delta \geq 0, \tag{2.3}$$

there exists a solution q of equation (2.2) and a constant $\gamma \geq 0$ dependent only on δ such that

$$\|f(x) - q(x)\| \leq \gamma \text{ for all } x \in G. \tag{2.4}$$

Proposition 2.1. *Let σ be an antiautomorphism of the semigroup G such that $\sigma \circ \sigma = I$. Let B a Banach space. Suppose that $f : G \longrightarrow B$ satisfies the inequality (2.3). Then for every $x \in G$, the limit*

$$g(x) = \lim_{n \to +\infty} 2^{-2n} \left[f(x^{2^n}) + \sum_{k=1}^{n} 2^{k-1} f((x^{2^{n-k}} \sigma(x)^{2^{n-k}})^{2^{k-1}}) \right] \tag{2.5}$$

exists. Moreover, $g : G \longrightarrow \mathbb{C}$ is a unique function satisfying

$$\|f(x) - g(x)\| \leq \delta, \text{ and } g(x^2) + g(x\sigma(x)) = 4g(x) \text{ for all } x \in G. \tag{2.6}$$

Proof. Assume that $f : G \longrightarrow \mathbb{C}$ satisfies the inequality (2.3) and define by induction the sequence function $f_0(x) = f(x)$ and $f_n(x) = \frac{1}{2}[f_{n-1}(x^2) + f_{n-1}(x\sigma(x))]$ for $n \geq 1$. By direct computation, we obtain

$$f_n(x) = 2^{-n} \left[f(x^{2^n}) + \sum_{k=1}^{n} 2^{k-1} f((x^{2^{n-k}} \sigma(x)^{2^{n-k}})^{2^{k-1}}) \right]$$

for all $n \geq 1$.

By letting $x = y$, in (2.3) we get

$$\left\| \frac{1}{2}[f(x^2) + f(x\sigma(x))] - 2f(x) \right\| \leq \delta, \tag{2.7}$$

so

$$\|f_1(x) - 2f_0(x)\| \leq \delta \text{ for all } x \in G. \tag{2.8}$$

In the following, we prove by induction the inequalities

$$\|f_n(x) - 2f_{n-1}(x)\| \leq \delta \tag{2.9}$$

$$\|f_n(x) - 2^n f(x)\| \leq (2^n - 1)\delta \tag{2.10}$$

for all $n \in \mathbb{N}$ and $x \in G$. It is clear that (2.8) is (2.9) for $n = 1$. The inductive step must now be demonstrated to hold true for the integer $n + 1$, that is

$$\begin{aligned}
(2.11)\quad \|f_{n+1}(x) - 2f_n(x)\| &= \left\|\frac{1}{2}[f_n(x^2) + f_n(x\sigma(x))] - 2\frac{1}{2}[f_{n-1}(x^2) + f_{n-1}(x\sigma(x))]\right\| \\
&\le \frac{1}{2}[\|f_n(x^2) - 2f_{n-1}(x^2)\|] + \left\|\frac{1}{2}[f_n(x\sigma(x)) - 2f_{n-1}(x\sigma(x))]\right\| \\
&\le \frac{1}{2}[\delta + \delta] = \delta.
\end{aligned}$$

This proves that (2.9) is true for any natural number n.

Now, by using the inequality

$$(2.12)\qquad \|f_n(x) - 2^n f(x)\| \le \|f_n(x) - 2f_{n-1}(x)\| + 2\|f_{n-1}(x) - 2^{n-1}f(x)\|$$

we check that (2.10) holds true for any $n \in \mathbb{N}$.

Let us define

$$(2.13)\qquad g_n(x) = \frac{f_n(x)}{2^n} = 2^{-2n}\left[f(x^{2^n}) + \sum_{k=1}^{n} 2^{k-1} f((x^{2^{n-k}}\sigma(x)^{2^{n-k}})^{2^{k-1}})\right]$$

for any positive integer n and $x \in G$.

Now, by using (2.11) and (2.13), we get

$$(2.14)\qquad \|g_{n+1}(x) - g_n(x)\| \le \frac{\delta}{2^{n+1}}.$$

It easily follows that $\{g_n(x)\}$ is a Cauchy sequence for all $x \in G$. Since B is complete, we can define $g(x) = \lim_{n\longrightarrow+\infty} g_n(x)$ for any $x \in G$. From (2.10), one can verify that g satisfies the first assertion of (2.6). Now, we will show that g satisfies the second assertion of (2.6). By induction one proves that the sequence $f_n(x)$ satisfies

$$(2.15)\qquad \left\|\frac{1}{2}[f_n(x^2) + f_n(x\sigma(x))] - f_n(x) - f_n(x)\right\| \le \delta$$

for all $n \in \mathbb{N}$.

For $n = 1$, we have

$$\begin{aligned}
(2.16)\quad &\left\|\frac{1}{2}[f_1(x^2) + f_1(x\sigma(x))] - f_1(x) - f_1(x)\right\| \\
&= \left\|\frac{1}{2}\left[\frac{1}{2}\left[f\left(x^{2^2}\right) + f(x^2\sigma(x)^2) + f((x\sigma(x))^2) + f((x\sigma(x))^2)\right]\right.\right. \\
&\qquad \left.\left. - \frac{1}{2}[f(x^2) + f(x\sigma(x))] - \frac{1}{2}[f(x^2) + f(x\sigma(x))]\right]\right\| \\
&\le \left\|\frac{1}{2}\left[\frac{1}{2}\left[f\left(x^{2^2}\right) + f(x^2\sigma(x)^2) - 2f(x^2)\right]\right]\right. \\
&\qquad \left. + \frac{1}{2}\left[\frac{1}{2}\left[f((x\sigma(x))^2) + f((x\sigma(x))^2) - 2f(x\sigma(x))\right]\right]\right\| \\
&\le \frac{1}{2}[\delta + \delta] = \delta.
\end{aligned}$$

So (2.15) is true for $n = 1$. We assume then that (2.15) holds for n and we prove that (2.15) is true for $n + 1$.

$$\begin{aligned}
(2.17)\qquad & \left\| \frac{1}{2}[f_{n+1}(x^2) + f_{n+1}(x\sigma(x))] - f_{n+1}(x) - f_{n+1}(x) \right\| \\
&= \left\| \frac{1}{2}\left[\frac{1}{2}\left[f_n\left(x^{2^2}\right) + f_n(x^2\sigma(x)^2) + f_n((x\sigma(x))^2) + f_n((x\sigma(x))^2)\right]\right.\right. \\
&\qquad \left.\left. -\frac{1}{2}[f_n(x^2) + f_n(x\sigma(x))] - \frac{1}{2}\left[f_n(x^2) + f_n(x\sigma(x))\right]\right]\right\| \\
&\le \left\| \frac{1}{2}\left[\frac{1}{2}\left[f_n\left(x^{2^2}\right) + f_n(x^2\sigma(x)^2) - 2f_n(x^2)\right]\right]\right\| \\
&\qquad + \left\| \frac{1}{2}\left[\frac{1}{2}[f_n((x\sigma(x))^2) + f_n((x\sigma(x))^2) - 2f_n(x\sigma(x))]\right]\right\| \\
&\le \frac{1}{2}[\delta + \delta] = \delta.
\end{aligned}$$

Consequently, the sequence $g_n(x)$ satisfies

$$(2.18)\qquad \left\| \frac{1}{2}[g_n(x^2) + g_n(x\sigma(x))] - g_n(x) - g_n(x) \right\| \le \frac{\delta}{2^n}$$

for all $n \in \mathbb{N}$ and then by letting $n \to +\infty$ we get the desired result.

Assume now that there exists another mapping $h : G \longrightarrow B$ which satisfies $\|f(x) - h(x)\| \le \delta$ and $h(x^2) + h(x\sigma(x)) = 4h(x)$ for all $x \in G$.

First, we will prove by mathematical induction that

$$(2.19)\qquad \|f_n(x) - 2^n h(x)\| \le \delta \quad \text{for all} \quad x \in G.$$

For $n = 1$, we have

$$\begin{aligned}
(2.20)\qquad \|f_1(x) - 2h(x)\| &= \left\| \frac{1}{2}[f(x^2) + f(x\sigma(x))] - \frac{1}{2}[h(x^2) + h(x\sigma(x))] \right\| \\
&\le \frac{1}{2}[\|f(x^2) - h(x^2)\| + \|f(x\sigma(x)) - h(x\sigma(x))\|] \\
&\le \frac{1}{2}[\delta + \delta] = \delta.
\end{aligned}$$

Suppose (2.19) is true for n and we will prove it for $n + 1$. Hence, we have

$$\begin{aligned}
(2.21)\qquad \|f_{n+1}(x) - 2^{n+1}h(x)\| &= \left\| \frac{1}{2}[f_n(x^2) + f_n(x\sigma(x))] - 2^n\frac{1}{2}[h(x^2) + h(x\sigma(x))] \right\| \\
&\le \frac{1}{2}[\|f_n(x^2) - 2^n h(x^2)\| + \|f_n(x\sigma(x)) - 2^n h(x\sigma(x))\|] \\
&\le \frac{1}{2}[\delta + \delta] = \delta.
\end{aligned}$$

This proves that (2.19) is true for all $n \in \mathbb{N}$. From (2.19), we obtain $\left\| \frac{f_n(x)}{2^n} - h(x) \right\| \le \frac{\delta}{2^n}$, so by letting $n \to +\infty$ and by using the definition of g we get $g = h$. This completes the proof of the theorem. □

By using the precedent proof we easily obtain the following result.

Proposition 2.2. *Let σ be an automorphism of the semigroup G such that $\sigma \circ \sigma = I$. Let B a Banach space. Suppose that $f : G \longrightarrow B$ satisfies the inequality (2.3). Then for every $x \in G$, the limit*

$$g(x) = \lim_{n \to +\infty} 2^{-n} f_n(x)$$

exists, and g is a unique function satisfying

$$\|f(x) - g(x)\| \leq \delta, \quad \text{and} \quad g(x^2) + g(x\sigma(x)) = 4g(x) \quad \text{for all } x \in G,$$

where the sequence of functions f_n is defined on G by the formulas $f_0(x) = f(x)$ and $f_n(x) = \frac{1}{2}[f_{n-1}(x^2) + f_{n-1}(x\sigma(x))]$ for $n \geq 1$.

The following result shows that in this context the only property of B (Definition 2.1) involved is the completeness. For the proof, we refer to the one used by Yang in [22].

Theorem 2.3. *Let σ be a morphism of the semigroup G such that $\sigma \circ \sigma = I$. Suppose that equation (2.2) is stable for the pair $(G, \mathbb{C})$ (resp. $(G, \mathbb{R})$) Then for every complex (resp. real) Banach space B, (2.2) is stable for the pair (G, B).*

The main result of the present section is the following

Theorem 2.4. *Let σ be an antiautomorphism of the amenable semigroup G such that $\sigma \circ \sigma = I$. Then equation (2.2) is stable for the pair $(G, \mathbb{C})$.*

First, we prove the following useful lemma

Lemma 2.5. *Let σ be an antiautomorphism of the semigroup G such that $\sigma \circ \sigma = I$. Let B be a Banach space. Suppose that $f : G \longrightarrow B$ satisfies the inequality*

$$\left\| \frac{1}{2}(f(xy) + f(x\sigma(y))) - f(x) - f(y) \right\| \leq \delta, \quad \text{for some } \delta \geq 0. \tag{2.22}$$

Then for every $x \in G$, the limit

$$q(x) = \lim_{n \to +\infty} 2^{-2n} \left\{ f(x^{2^n}) + (2^n - 1) f(x^{2^{n-1}} \sigma(x)^{2^{n-1}}) \right\} \tag{2.23}$$

exists. Moreover, the mapping q satisfies the inequality

$$\| \; f(x) - q(x) \; \| \leq 7\delta \quad \text{for all } x \in G. \tag{2.24}$$

Proof. Assume that $f : G \longrightarrow B$ satisfies the inequality (2.22). We will prove by induction that

$$\begin{aligned} &\left\| f(x) - \frac{1}{2^{2n}} \{ f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x)) \} \right\| \\ &\qquad\qquad \leq 2\left(\frac{7}{2} + \frac{3}{2^{2n-1}} - \frac{19}{2^{n+1}} \right) \delta \end{aligned} \tag{2.25}$$

for some positive integer n. By letting $y = x$, in (2.22) we get

$$\|f(x^2) + (2-1)f(x\sigma(x)) - 2^2 f(x)\| \leq 2\delta, \tag{2.26}$$

so

$$\left\| f(x) - \frac{1}{2^2} \{ f(x^2) + (2-1) f(x\sigma(x)) \} \right\| \leq \delta \left(1 - \frac{1}{2} \right) \quad \text{for all } \; x \in G. \tag{2.27}$$

This proves (2.25) for $n = 1$. The inductive step must now be demonstrated to hold true for the integer $n + 1$, that is

$$\begin{aligned}
&\left\| f(x) - \frac{1}{2^{2(n+1)}} \left\{ f(x^{2^{n+1}}) + (2^{n+1} - 1) f(x^{2^n}\sigma(x)^{2^n}) \right\} \right\| \\
&\le \frac{1}{2^{2(n+1)}} \left\| f(x^{2^{n+1}}) + f\left(x^{2^n}\sigma(x)^{2^n}\right) - 4f(x^{2^n}) \right\| \\
&\quad + \frac{1}{2^{2(n+1)}} \left\| 2(2^n - 1) f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}} x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) - 4(2^n - 1) f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) \right\| \\
&\quad + \frac{1}{2^{2(n+1)}} \left\| 4f(x^{2^n}) + 4(2^n - 1) f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) - 2^{2(n+1)} f(x) \right\| \\
&\quad + \frac{2(2^n - 1)}{2^{2(n+1)}} \left\| f\left(x^{2^n}\sigma(x)^{2^n}\right) - f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}} x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) \right\| \\
&\le \frac{2\delta}{2^{2(n+1)}} + \frac{(2^n - 1)2\delta}{2^{2(n+1)}} + \left(\frac{7}{2} + \frac{3}{2^{2n-1}} - \frac{19}{2^{n+1}} \right) 2\delta \\
&\quad + \frac{2(2^n - 1)}{2^{2(n+1)}} \left\| f(x^{2^n}\sigma(x)^{2^n}) - f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}} x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) \right\|.
\end{aligned}$$

To complete the proof of the induction assumption (2.25), we need the following inequalities. Let $x = y = e$ in (2.22) to get $\|2f(e)\| \le 2\delta$. Putting $x = e$ in (2.22), gives

$$\| f(y) + f(\sigma(y)) - 2f(e) - 2f(y)\| \le 2\delta.$$

Consequently,

$$\|f(y) - f(\sigma(y))\| \le 4\delta. \tag{2.28}$$

By interchanging x by y in (2.22), we obtain

$$\|f(yx) + f(y\sigma(x)) - 2f(x) - 2f(y)\| \le 2\delta. \tag{2.29}$$

By using (2.22), (2.28), (2.29) and the triangle inequality, we deduce that

$$\|f(xy) - f(yx)\| \le 8\delta. \tag{2.30}$$

Now, from (2.22), we obtain

$$\left\| 2f\left(x^{2^{n-1}}\sigma(x)^{2^n} x^{2^{n-1}}\right) - 2f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) - 2f\left(\sigma(x)^{2^{n-1}} x^{2^{n-1}}\right) \right\| \le 2\delta. \tag{2.31}$$

Since

$$\left\| f\left(x^{2^n}\sigma(x)^{2^n}\right) - f\left(\sigma(x)^{2^n} x^{2^n}\right) \right\| \le 8\delta \tag{2.32}$$

and

$$\left\| f\left(x^{2^{n-1}}\sigma(x)^{2^n} x^{2^{n-1}}\right) - f(x^{2^n}\sigma(x)^{2^n}) \right\| \le 8\delta \tag{2.33}$$

then

$$\left\| 2f\left(x^{2^{n-1}}\sigma(x)^{2^n} x^{2^{n-1}}\right) - 4f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right) \right\| \le 18\delta \tag{2.34}$$

and

$$\begin{aligned}(2.35)\qquad &\left\|f(x^{2^n}\sigma(x)^{2^n})-2f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq \left\|f\left(x^{2^{n-1}}\sigma(x)^{2^n}x^{2^{n-1}}\right)-f(x^{2^n}\sigma(x)^{2^n})\right\|\\ &\qquad+\left\|f\left(x^{2^{n-1}}\sigma(x)^{2^n}x^{2^{n-1}}\right)-f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq 8\delta+\frac{18\delta}{2}=17\delta.\end{aligned}$$

Finally, we deduce that

$$\begin{aligned}(2.36)\qquad &\left\|f(x^{2^n}\sigma(x)^{2^n})-f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq \left\|f(x^{2^n}\sigma(x)^{2^n})-2f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\qquad+\left\|2f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)-f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq 17\delta+\delta=18\delta\end{aligned}$$

and consequently,

$$\begin{aligned}&\left\|f(x)-\frac{1}{2^{2(n+1)}}\left\{f\left(x^{2^{n+1}}\right)+(2^{n+1}-1)f\left(x^{2^n}\sigma(x)^{2^n}\right)\right\}\right\|\\ &\leq \frac{2\delta}{2^{2(n+1)}}+\frac{2(2^n-1)\delta}{2^{2(n+1)}}+\left(\frac{7}{2}+\frac{3}{2^{2n-1}}-\frac{19}{2^{n+1}}\right)2\delta+\frac{2(2^n-1)}{2^{2(n+1)}}18\delta\\ &=2\left(\frac{7}{2}+\frac{3}{2^{2n+1}}-\frac{19}{2^{n+2}}\right)\delta.\end{aligned}$$

This proves the validity of the inequality (2.25).

Let us define

$$(2.37)\qquad q_n(x)=\frac{1}{2^{2n}}\left\{f(x^{2^n})+(2^n-1)f(x^{2^{n-1}}\sigma(x)^{2^{n-1}})\right\}$$

for any positive integer n and $x\in G$.

Then $\{q_n(x)\}$ is a Cauchy sequence for every $x\in G$. In fact by using (2.22), (2.36) and (2.37), we get

$$\begin{aligned}&\|q_{n+1}(x)-q_n(x)\|\\ &\leq \frac{1}{2^{2(n+1)}}\left\|f\left(x^{2^{n+1}}\right)+f(x^{2^n}\sigma(x)^{2^n})-4f(x^{2^n})\right\|\\ &\qquad+\frac{1}{2^{2(n+1)}}\left\|2(2^n-1)f\left(x^{2^n}\sigma(x)^{2^n}\right)-4(2^n-1)f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq \frac{\delta}{2^{2(n+1)}}+\frac{1}{2^{2(n+1)}}\Big\|2(2^n-1)f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\\ &\qquad-4(2^n-1)f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\Big\|\\ &\qquad+\frac{2(2^n-1)}{2^{2(n+1)}}\left\|f(x^{2^n}\sigma(x)^{2^n})-f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\|\\ &\leq \frac{2\delta}{2^{2(n+1)}}+\frac{2(2^n-1)\delta}{2^{2(n+1)}}+\frac{36\delta(2^n-1)}{2^{2(n+1)}}\\ &\leq \frac{40\delta}{2^n}.\end{aligned}$$

It easily follows that $\{q_n(x)\}$ is a Cauchy sequence for all $x \in G$. Since B is complete, we can define $q(x) = \lim_{n\longrightarrow+\infty} q_n(x)$ for any $x \in G$ and, in view of (2.25) one can verify that q satisfies the inequality (2.24). This completes the proof of Lemma 2.5. □

Proof of Theorem 2.4. We follow the ideas and the computations used in [22]. By using (2.30) one derives the functional inequality

$$|f((xy)^n) - f((yx)^n)| \leq 8\delta. \tag{2.38}$$

In addition, from (2.3), (2.38) and the triangle inequality we deduce that

$$\begin{aligned} &|f((xy)^{2^n}(\sigma(xy))^{2^n}) - f((yx)^{2^n}(\sigma(yx))^{2^n})| \\ &\quad\leq |f((xy)^{2^n}(\sigma(xy))^{2^n}) + f((xy)^{2^n}(xy)^{2^n}) - 4f((xy)^{2^n})| \\ &\qquad + |-f((yx)^{2^n}(yx)^{2^n}) - f((yx)^{2^n}(\sigma(yx))^{2^n}) + 4f((yx)^{2^n})| \\ &\qquad + \left|f((xy)^{2^n}(xy)^{2^n}) - f((yx)^{2^n}(yx)^{2^n})\right| + 4\left|f((xy)^{2^n}) - f((yx)^{2^n})\right| \\ &\quad\leq 2\delta + 2\delta + 8\delta + 32\delta = 44\delta. \end{aligned} \tag{2.39}$$

From Lemma 2.5, for every $x \in G$, the limit

$$q(x) = \lim_{n\to+\infty} 2^{-2n}\left\{f(x^{2^n}) + (2^n - 1)f\left(x^{2^{n-1}}\sigma(x)^{2^{n-1}}\right)\right\} \tag{2.40}$$

exists and

$$|f(x) - q(x)| \leq 7\delta. \tag{2.41}$$

Furthermore, in view of (2.38) – (2.39) q satisfies the relation

$$q(xy) = q(yx), \quad x, y \in G \tag{2.42}$$

and by (2.41) – (2.22) q satisfies the inequality equation

$$|q(xy) + q(x\sigma(y)) - 2q(x) - 2q(y)| \leq 44\delta \tag{2.43}$$

for all $x, y \in G$.

Consequently, for any fixed $y \in G$ the function

$$x \longmapsto q(xy) + q(x\sigma(y)) - 2q(x)$$

is bounded. Since G is amenable, there exists an invariant mean m_x on the space of bounded, complex-functions on G. With the help of m_x we define the following function on G

$$\psi(y) = m_x\{q_y + q_{\sigma(y)} - 2q\} \tag{2.44}$$

for all $y \in G$, where $q_y(z) = q(zy)$, $z \in G$.

Furthermore, by using (2.29) and (2.44), we get

$$\begin{aligned} \psi(zy) + \psi(\sigma(z)y) &= m_x\{q_{zy} + q_{\sigma(y)\sigma(z)} - 2q\} + m_x\{q_{\sigma(z)y} + q_{\sigma(y)z} - 2q\} \\ &= m_x\{{}_{zy}q + q_{\sigma(y)\sigma(z)} - 2q\} + m_x\{{}_{\sigma(z)y}q + q_{\sigma(y)z} - 2q\} \\ &= m_x\{{}_{zy}q +{}_{\sigma(z)y} q - 2({}_yq)\} + m_x\{q_{\sigma(y)\sigma(z)} + q_{\sigma(y)z} - 2q_{\sigma(y)}\} \\ &\qquad + m_x\{2q_y + 2q_{\sigma(y)} - 4q\} \\ &= 2\psi(z) + 2\psi(y). \end{aligned} \tag{2.45}$$

So, $Q(y) = \frac{\psi(y)}{2}$ satisfies equation (2.2) and the following inequality

$$\begin{aligned} |Q(y) - q(y)| &= \frac{1}{2}|m_x\{q_y + q_{\sigma(y)} - 2q - 2q(y)\}| \\ &\leq \sup_{x\in G} \frac{1}{2}|\{q(xy) + q(x\sigma(y)) - 2q(x) - 2q(y)\}| \\ &\leq \frac{44}{2}\delta = 22\delta. \end{aligned} \tag{2.46}$$

Consequently, there exists a mapping Q which satisfies the functional equation (2.2) and the inequality $|f(y) - Q(y)| \leq 29\delta$. This completes the proof of the theorem. □

Theorem 2.6. *Let σ be an automorphism of the amenable semigroup G such that $\sigma \circ \sigma = I$. Then equation (2.2) is stable for the pair $(G, \mathbb{C})$.*

Proof. From inequality (2.3), we deduce that for any fixed $y \in G$ the function $x \longmapsto f(xy) + f(x\sigma(y)) - 2f(x)$ is bounded. Since G is amenable, then we can define

$$\phi(y) = m_x\{f_y + f_{\sigma(y)} - 2f\} \tag{2.47}$$

for all $y \in G$.

We have

$$\begin{aligned} &\phi(yz)+\phi(y\sigma(z)) \\ &\quad= m_x\{f_{yz} + f_{\sigma(y)\sigma(z)} - 2f\} + m_x\{f_{y\sigma(z)} + f_{\sigma(y)z} - 2f\} \\ &\quad= m_x\{f_{yz} + f_{y\sigma(z)} - 2f_y\} + m_x\{f_{\sigma(y)\sigma(z)} + f_{\sigma(y)z} - 2f_{\sigma(y)}\} \\ &\qquad + 2m_x\{f_y + f_{\sigma(y)} - 2f\} \\ &\quad= \phi(z) + \phi(\sigma(z)) + 2\phi(y) = 2\phi(z) + 2\phi(y), \end{aligned} \tag{2.48}$$

so ϕ is a solution of equation (2.2). Moreover, we have

$$\begin{aligned} \left|f(y) - \frac{\phi(y)}{2}\right| &= \frac{1}{2}|m_x\{f_y + f_{\sigma(y)} - 2f - 2f(y)\}| \\ &\leq \frac{1}{2}\sup_{x\in G}|f(xy) + f(x\sigma(y)) - 2f(x) - 2f(y)| \leq \delta. \end{aligned} \tag{2.49}$$

This completes the proof of theorem. □

By using Theorem 2.4 and the proof of Proposition 2.1 we get the following corollaries.

In the first corollary, the estimate improves the ones obtained in the proof of Theorem 2.4.

Corollary 2.7. *Let σ be an antiautomorphism of the amenable semigroup G such that $\sigma\circ\sigma = I$. Let B a Banach space. Suppose that $f : G \longrightarrow B$ satisfies the inequality (2.3). Then for every $x \in G$, the limit*

$$Q(x) = \lim_{n\to+\infty} 2^{-2n}\left[f(x^{2^n}) + \sum_{k=1}^{n} 2^{k-1} f\left((x^{2^{n-k}}\sigma(x)^{2^{n-k}})^{2^{k-1}}\right)\right] \tag{2.50}$$

exists. Moreover, Q is the unique solution of equation (1.4) satisfying

$$\|f(x) - Q(x)\| \leq \delta \text{ for all } x \in G. \tag{2.51}$$

Corollary 2.8. *Let σ be a morphism of the amenable semigroup G such that $\sigma \circ \sigma = I$. Then for every Banach space B, equation (2.2) is stable for the pair (G, B).*

Corollary 2.9 ([20]). *Let $\sigma = I$. Let G be an amenable semigroup. Then for every Banach space B, equation*

$$f(xy) = f(x) + f(y), \quad x, y \in G \tag{2.52}$$

is stable for the pair (G, B).

Corollary 2.10 ([22]). *Let $\sigma(x) = x^{-1}$. Let G be an amenable group. Then for every Banach space B, equation*

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad x, y \in G \tag{2.53}$$

is stable for the pair (G, B).

Corollary 2.11 ([2]). *Let σ be an automorphism of the vector space G such that $\sigma \circ \sigma = I$. Then for every Banach space B, the equation*

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G \tag{2.54}$$

is stable for the pair (G, B).

3. Stability of Equation (1.5) in Amenable Semigroups

In this section we investigate the Hyers-Ulam stability of the functional equation

$$f(xy) + g(x\sigma(y)) = f(x) + g(y), \quad x, y \in G, \tag{3.1}$$

where G is an amenable semigroup with element unity e and $\sigma : G \longrightarrow G$ is an automorphism of G such that $\sigma \circ \sigma = I$.

The stability of equation (3.1) was studied by several authors in the case where G is an abelian group and $\sigma = -I$. For more information, see for example [12].

First we establish some results which will be instrumental in proving our main results.

In the following lemma, we will present a Hyers-Ulam stability result for Jensen's functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x), \quad x, y \in G. \tag{3.2}$$

Lemma 3.1. *Let G be an amenable semigroup. Let σ be an homomorphism of G such that $\sigma \circ \sigma = I$ and let $f : G \longrightarrow \mathbb{C}$ be a function. Assume that there exists $\delta \geq 0$ such that*

$$|f(xy) + f(x\sigma(y)) - 2f(x)| \leq \delta \tag{3.3}$$

for all $x, y \in G$. Then, there exists a solution $J : G \longrightarrow \mathbb{C}$ of Jensen's functional equation (3.2) such that

$$|f(x) - J(x) - f(e)| \leq \delta \tag{3.4}$$

for all $x \in G$.

Proof. Let us denote by $f^e(x) = \frac{f(x)+f(\sigma(x))}{2}$ the even part of f and by $f^o(x) = \frac{f(x)-f(\sigma(x))}{2}$ the odd part of f.

By replacing x by $\sigma(x)$ and y by $\sigma(y)$ in (3.3), we get

$$|f(\sigma(x)\sigma(y)) + f(\sigma(x)y) - 2f(\sigma(x))| \leq \delta. \tag{3.5}$$

Now, if we add (subtract) the argument of the inequality (3.3) to (from) inequality (3.5), we deduce that the functions f^e and f^o satisfy the following inequalities

$$|f^e(xy) + f^e(x\sigma(y)) - 2f^e(x)| \leq \delta \tag{3.6}$$

$$|f^o(xy) + f^o(x\sigma(y)) - 2f^o(x)| \leq \delta \tag{3.7}$$

for all $x, y \in G$.

By putting $x = e$ in (3.6), we obtain

$$|f^e(y) - f(e)| \leq \frac{\delta}{2}. \tag{3.8}$$

The inequality (3.7) can be written as follows

$$|f^o(yx) - f^o(\sigma(y)x) - 2f^o(y)| \leq \delta. \tag{3.9}$$

This implies that for fixed $y \in G$, the function $x \longmapsto f^o(yx) - f^o(\sigma(y)x)$ is bounded. Since G is amenable, let m_x be an invariant mean on the space of complex bounded functions on G and define the mapping:

$$\psi(y) = m_x\{{}_y f^o - {}_{\sigma(y)} f^o\} \quad \text{for all } y \in G. \tag{3.10}$$

Consequently from (3.10), we obtain that ψ satisfies the Jensen's functional equation

$$\begin{aligned} \psi(yz)+\psi(y\sigma(z)) &= m_x\{{}_{yz}f^o - {}_{\sigma(y)\sigma(z)} f^o\} + m_x\{{}_{y\sigma(z)}f^o - {}_{\sigma(y)z} f^o\} \\ &= m_x\{{}_{yz}f^o - {}_{\sigma(y)z} f^o\} + m_x\{{}_{y\sigma(z)}f^o - {}_{\sigma(y)\sigma(z)} f^o\} \\ &= m_x\{{}_z[{}_y f^o - {}_{\sigma(y)} f^o]\} + m_x\{{}_{\sigma(z)}[{}_y f^o - {}_{\sigma(y)} f^o]\} \\ &= 2\psi(y). \end{aligned} \tag{3.11}$$

The function $J(y) = \frac{\psi(y)}{2}$ satisfies the Jensen's functional equation (3.2) and the following inequality

$$|J(y) - f^o(y)| \leq \frac{1}{2} \sup_{x \in G} |f^o(yx) - f^o(\sigma(y)x) - 2f^o(y)| \leq \frac{\delta}{2}. \tag{3.12}$$

Finally, we obtain

$$\begin{aligned} |f(y) - J(y) - f(e)| &= |f^e(y) + f^o(y) - J(y) - f(e)| \\ &\leq |f^e(y) - f(e)| + |f^o(y) - J(y)| \leq \delta. \end{aligned} \tag{3.13}$$

This completes the proof of Lemma 3.1. □

By using the proof of the preceding lemma, we get the stability of the Jensen function equation

$$f(yx) + f(\sigma(y)x) = 2f(x), \quad x, y \in G. \tag{3.14}$$

Lemma 3.2. *Let G be an amenable semigroup. Let σ be a homomorphism of G such that $\sigma \circ \sigma = I$ and let $f : G \longrightarrow \mathbb{C}$ be a function. Assume that there exists $\delta \geq 0$ such that*

$$|f(yx) + f(\sigma(y)x) - 2f(x)| \leq \delta \tag{3.15}$$

for all $x, y \in G$. Then, there exists a solution $J : G \longrightarrow \mathbb{C}$ of Jensen's functional equation (3.14) such that

$$|f(x) - J(x) - f(e)| \leq \delta \tag{3.16}$$

for all $x \in G$. More precisely, J is given by the formula

$$J(y) = m_x\{f^o_y - f^o_{\sigma(y)}\} \text{ for all } y \in G. \tag{3.17}$$

In the following lemma, we obtain a partial stability theorem for the Pexider's functional equation

$$f_1(xy) + f_2(x\sigma(y)) = f_3(x) + f_4(y), \quad x, y \in G \tag{3.18}$$

that includes the functional equation (1.5) and the Drygas's functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in G \tag{3.19}$$

as special cases.

Lemma 3.3. *Let G be an amenable semigroup. Let σ be an automorphism of G such that $\sigma \circ \sigma = I$. If the functions $f_1, f_2, f_3, f_4 : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|f_1(xy) + f_2(x\sigma(y)) - f_3(x) - f_4(y)| \leq \delta \tag{3.20}$$

for all $x, y \in G$, then there exists a unique function $q : G \longrightarrow \mathbb{C}$, a solution of equation (1.4). Also, there exists a solution J_1, (resp. J_2): $G \longrightarrow \mathbb{C}$ of Jensen's functional equation (3.14), (resp. (3.2)) such that,

$$|f_3(x) - J_2(x) - q(x) - f_3(e)| \leq 16\delta, \tag{3.21}$$

$$|f_4(x) - J_1(x) - q(x) - f_4(e)| \leq 16\delta, \tag{3.22}$$

$$\left| f_1^e(x) + f_2^e(x) - q(x) - \frac{1}{2} f_1(e) - \frac{1}{2} f_2(e) \right| \leq 6\delta, \tag{3.23}$$

$$|(f_1^e - f_2^e)(xy) - (f_1^e - f_2^e)(x\sigma(y))| \leq 12\delta, \tag{3.24}$$

$$\left| f_1^o(x) - \frac{1}{2} J_1(x) - \frac{1}{2} J_2(x) \right| \leq 10\delta$$

and

$$\left| f_2^o(x) - \frac{1}{2} J_2(x) + \frac{1}{2} J_1(x) \right| \leq 10\delta$$

for all $x, y \in G$.

Proof. In the present proof, we follow the computations used in the papers [1], [12], and [23].

For any function $f : G \longrightarrow \mathbb{C}$, we define $F(x) = f(x) - f(e)$.

By putting $x = y = e$ in (3.20), we get

$$|f_1(e) + f_2(e) - f_3(e) - f_4(e)| \leq \delta. \tag{3.25}$$

Consequently, if we subtract the inequality (3.20) from the new inequality (3.25), we obtain

$$|F_1(xy) + F_2(x\sigma(y)) - F_3(x) - F_4(y)| \leq 2\delta. \tag{3.26}$$

Now, by replacing x by $\sigma(x)$ and y by $\sigma(y)$ in (3.26) and if we add (subtract) the inequality obtained to (3.26), we deduce that

$$|F_1^e(xy) + F_2^e(x\sigma(y)) - F_3^e(x) - F_4^e(y)| \leq 2\delta, \tag{3.27}$$

and

$$|F_1^o(xy) + F_2^o(x\sigma(y)) - F_3^o(x) - F_4^o(y)| \leq 2\delta \tag{3.28}$$

for all $x, y \in G$. Hence, if we replace y by e, and x by e respectively in (3.27), we get

$$|F_1^e(x) + F_2^e(x) - F_3^e(x)| \leq 2\delta \tag{3.29}$$

and

$$|F_1^e(y) + F_2^e(y) - F_4^e(y)| \leq 2\delta. \tag{3.30}$$

So, in view of (3.27), (3.29) and (3.30), we obtain

$$\begin{aligned} &|F_1^e(xy) + F_2^e(x\sigma(y)) - (F_1^e + F_2^e)(x) - (F_1^e + F_2^e)(y)| \\ &\quad \leq |F_1^e(xy) + F_2^e(x\sigma(y)) - F_3^e(x) - F_4^e(y)| \\ &\qquad + |F_1^e(x) + F_2^e(x) - F_3^e(x)| + |F_1^e(y) + F_2^e(y) - F_4^e(y)| \\ &\quad \leq 6\delta. \end{aligned} \tag{3.31}$$

By replacing y by $\sigma(y)$ in (3.31), we get the following

$$|F_1^e(x\sigma(y)) + F_2^e(xy) - (F_1^e + F_2^e)(x) - (F_1^e + F_2^e)(y)| \leq 6\delta. \tag{3.32}$$

If we add (subtract) the inequality (3.31) to (3.32), we get

$$|(F_1^e + F_2^e)(xy) + (F_1^e + F_2^e)(x\sigma(y)) - 2(F_1^e + F_2^e)(x) - 2(F_1^e + F_2^e)(y)| \leq 12\delta, \tag{3.33}$$

$$|(F_1^e - F_2^e)(xy) - (F_1^e - F_2^e)(x\sigma(y))| \leq 12\delta \tag{3.34}$$

for all $x, y \in E_1$. Hence, in view of Theorem 2.6, there exists a unique function q, a solution of equation (1.4) such that

$$|(F_1^e + F_2^e)(x) - q(x)| \leq 6\delta \text{ for all } x \in G. \tag{3.35}$$

Consequently, from (3.29), (3.30) and (3.35), we deduce that

$$|F_3^e(x) - q(x)| \leq 8\delta \tag{3.36}$$

and

$$|F_4^e(x) - q(x)| \leq 8\delta \tag{3.37}$$

for all $x \in G$.

On the other hand, from (3.28) we get

$$|F_3^o(x) - F_1^o(x) - F_2^o(x)| \leq 2\delta \tag{3.38}$$

and

$$|F_4^o(x) - F_1^o(x) + F_2^o(x)| \leq 2\delta, \tag{3.39}$$

for all $x \in G$. Hence, we obtain

$$|2F_1^o(x) - F_3^o(x) - F_4^o(x)| \leq 4\delta \tag{3.40}$$

and

$$|2F_2^o(x) - F_3^o(x) + F_4^o(x)| \leq 4\delta \tag{3.41}$$

for all $x \in G$ and consequently, we have

$$\begin{aligned} &|F_3^o(xy) + F_3^o(x\sigma(y)) - 2F_3^o(x)| \\ &\quad \leq |F_3^o(xy) - F_1^o(xy) - F_2^o(xy)| \\ &\qquad + |F_3^o(x\sigma(y)) - F_1^o(x\sigma(y)) - F_2^o(x\sigma(y))| \\ &\qquad + |F_1^o(xy) + F_2^o(x\sigma(y)) - F_3^o(x) - F_4^o(y)| \\ &\qquad + |F_1^o(x\sigma(y)) + F_2^o(xy) - F_3^o(x) - F_4^o(\sigma(y))| \\ &\quad \leq 8\delta \end{aligned} \tag{3.42}$$

and

$$|F_4^o(yx) + F_4^o(\sigma(y)x) - 2F_4^o(x)| \tag{3.43}$$
$$\leq |F_4^o(yx) - F_1^o(yx) + F_2^o(yx)|$$
$$+ |F_4^o(\sigma(y)x) - F_1^o(\sigma(y)x) + F_2^o(\sigma(y)x)|$$
$$+ |F_1^o(yx) + F_2^o(y\sigma(x)) - F_3^o(y) - F_4^o(x)|$$
$$+ |F_1^o(\sigma(y)x) + F_2^o(\sigma(y)\sigma(x)) - F_3^o(\sigma(y)) - F_4^o(x)|$$
$$\leq 8\delta$$

for all $x, y \in G$.

Now, from Lemma 3.1 and Lemma 3.2 there exist two solutions of Jensen's functional equation (3.14) and (3.2), $J_1, J_2 : G \longrightarrow \mathbb{C}$ such that

$$|F_4^o(x) - J_1(x)| \leq 8\delta. \tag{3.44}$$

and

$$|F_3^o(x) - J_2(x)| \leq 8\delta \tag{3.45}$$

for all $x \in G$. Now, by small computations, we obtain the rest of the proof. □

By using the previous lemmas, we may deduce our main result.

Theorem 3.4. *Let G be an amenable semigroup. Let σ be an automorphism of G such that $\sigma \circ \sigma = I$. If the functions $f, g : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|f(xy) + g(x\sigma(y)) - f(x) - g(y)| \leq \delta, \tag{3.46}$$

for all $x, y \in G$, then there exists a unique function $q : G \longrightarrow \mathbb{C}$, a solution of equation (1.4). Also, there exist two solutions J_1, (resp. J_2): $G \longrightarrow \mathbb{C}$ of Jensen's functional equation (3.14), (resp. (3.2)) such that

$$|f(x) - J_2(x) - q(x) - f(e)| \leq 16\delta \tag{3.47}$$

and

$$|g(x) - J_1(x) - q(x) - g(e)| \leq 16\delta, \tag{3.48}$$

for all $x \in G$.

The stability of the Drygas's functional equation (3.19) is a consequence of the preceding theorem.

Theorem 3.5. *Let G be an amenable semigroup. Let σ be an automorphism of G such that $\sigma \circ \sigma = I$. Let the function $f : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x) - f(y) - f(\sigma(y))| \leq \delta, \tag{3.49}$$

for all $x, y \in G$. Then there exists a unique function $q : G \longrightarrow \mathbb{C}$, a solution of equation (1.4), and a solution $J : G \longrightarrow \mathbb{C}$ of Jensen's functional equation (3.2) such that

$$|f(x) - J(x) - q(x) - f(e)| \leq 16\delta \tag{3.50}$$

for all $x \in G$.

Corollary 3.6. *Let G be an amenable semigroup with a unity element. Let σ be an automorphism of G such that $\sigma \circ \sigma = I$. If the functions $f_1, f_2, f_3, f_4 : G \longrightarrow \mathbb{C}$ satisfy the functional equation*

$$f_1(xy) + f_2(x\sigma(y)) = f_3(x) + f_4(y) \tag{3.51}$$

for all $x, y \in G$, then there exists a quadratic function $q : G \longrightarrow \mathbb{C}$. There also exists a function $\nu : G \longrightarrow \mathbb{C}$, a solution of

$$\nu(xy) = \nu(x\sigma(y)), \quad x, y \in G. \tag{3.52}$$

In addition, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and two solutions J_1, (resp. J_2) of Jensen's equation (3.14) (resp. (3.2)) such that

$$f_1(x) = \frac{1}{2}J_1(x) + \frac{1}{2}J_2(x) + \frac{1}{2}\nu(x) + \frac{1}{2}q(x) + \alpha, \tag{3.53}$$

$$f_2(x) = -\frac{1}{2}J_1(x) + \frac{1}{2}J_2(x) - \frac{1}{2}\nu(x) + \frac{1}{2}q(x) + \beta, \tag{3.54}$$

$$f_3(x) = J_2(x) + q(x) + \gamma \tag{3.55}$$

and

$$f_4(x) = J_1(x) + q(x) + \delta \tag{3.56}$$

for all $x \in G$.

From Lemma 3.3, we can deduce the results obtained in [12].

Corollary 3.7. *Let G be a vector space. If the functions $f_1, f_2, f_3, f_4 : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|f_1(x + y) + f_2(x - y) - f_3(x) - f_4(y)| \leq \delta \tag{3.57}$$

for all $x, y \in G$, then there exists a unique function $q : G \longrightarrow \mathbb{C}$, a solution of equation (1.2). Also, $\alpha \in \mathbb{C}$ exists, and there are exactly two additive functions $a_1, a_2 : G \longrightarrow \mathbb{C}$ such that

$$\left| f_1(x) - \frac{1}{2}a_1(x) - \frac{1}{2}a_2(x) - \frac{1}{2}q(x) - f_1(0) - \alpha \right| \leq 19\delta, \tag{3.58}$$

$$\left| f_2(x) + \frac{1}{2}a_1(x) - \frac{1}{2}a_2(x) - \frac{1}{2}q(x) - f_2(0) + \alpha \right| \leq 19\delta, \tag{3.59}$$

$$|f_3(x) - a_2(x) - q(x) - f_3(0)| \leq 16\delta \tag{3.60}$$

and

$$|f_4(x) - a_1(x) - q(x) - f_4(0)| \leq 16\delta \tag{3.61}$$

for all $x \in G$.

The following corollary follows from Lemma 3.3. This result is well known in the commutative case, see for example [15].

Corollary 3.8. *Let G be an amenable semigroup. If the functions $f_1, f_2, f_3 : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|f_1(xy) - f_2(x) - f_3(y)| \leq \delta \tag{3.62}$$

for all $x, y \in G$, then there exists a unique additive function $a : G \longrightarrow \mathbb{C}$ such that

$$|f_1(x) - a(x) - f_1(e)| \leq 38\delta, \tag{3.63}$$

$$|f_2(x) - a(x) - f_2(e)| \leq 16\delta \tag{3.64}$$

and

$$|f_3(x) - a(x) - f_3(e)| \leq 16\delta \tag{3.65}$$

for all $x \in G$.

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