Journal of Inequalities in Pure and Applied Mathematics

GENERALIZED RELATIVE INFORMATION AND INFORMATION INEQUALITIES

INDER JEET TANEJA

Departamento de Matemática
Universidade Federal de Santa Catarina
88.040-900 Florianópolis,
SC, Brazil
EMail: taneja@mtm.ufsc.br
URL: http://www.mtm.ufsc.br/∼taneja

volume 5, issue 1, article 21,
2004.

Received 10 June, 2003;
accepted 01 August, 2003.

Communicated by: S.S. Dragomir

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
078-03

Abstract

In this paper, we have obtained bounds on Csiszár's *f-divergence* in terms of *relative information of type s* using Dragomir's [9] approach. The results obtained in particular lead us to bounds in terms of χ^2-*Divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination.*

2000 Mathematics Subject Classification: 94A17; 26D15.
Key words: Relative information; Csiszár's $f-$divergence; χ^2-divergence; Hellinger's discrimination; Relative information of type s; Information inequalities.

The author is thankful to the referee for valuable comments and suggestions on an earlier version of the paper.

Contents

1. Introduction

Let

$$\Delta_n = \left\{ P = (p_1, p_2, \dots, p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2,$$

be the set of complete finite discrete probability distributions.

The Kullback Leibler's [13] *relative information* is given by

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln \left(\frac{p_i}{q_i} \right), \tag{1.1}$$

for all $P, Q \in \Delta_n$.

In Δ_n, we have taken all $p_i > 0$. If we take $p_i \geq 0, \forall i = 1, 2, \dots, n$, then in this case we have to suppose that $0 \ln 0 = 0 \ln \left(\frac{0}{0}\right) = 0$. From the *information theoretic* point of view we generally take all the logarithms with base 2, but here we have taken only natural logarithms.

We observe that the measure (1.1) is not symmetric in P and Q. Its symmetric version, famous as *J-divergence* (Jeffreys [12]; Kullback and Leiber [13]), is given by

$$J(P||Q) = K(P||Q) + K(Q||P) = \sum_{i=1}^{n} (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right). \tag{1.2}$$

Let us consider the one parametric generalization of the measure (1.1), called *relative information of type* s given by

$$K_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1 \right], \quad s \neq 0, 1. \tag{1.3}$$

In this case we have the following limiting cases

$$\lim_{s \to 1} K_s(P||Q) = K(P||Q),$$

and

$$\lim_{s \to 0} K_s(P||Q) = K(Q||P).$$

The expression (1.3) has been studied by Vajda [22]. Previous to it many authors studied its characterizations and applications (ref. Taneja [20] and on line book Taneja [21]).

We have some interesting particular cases of the measure (1.3).

(i) When $s = \frac{1}{2}$, we have

$$K_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$$

where

$$B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}, \tag{1.4}$$

is the famous as Bhattacharya's [1] *distance*, and

$$h(P||Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2, \tag{1.5}$$

is famous as Hellinger's [11] *discrimination*.

(ii) When $s = 2$, we have

$$K_2(P||Q) = \frac{1}{2}\chi^2(P||Q),$$

where

(1.6) $$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1,$$

is the χ^2−*divergence* (Pearson [16]).

(iii) When $s = -1$, we have

$$K_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P),$$

where

(1.7) $$\chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i} = \sum_{i=1}^{n} \frac{q_i^2}{p_i} - 1.$$

For simplicity, let us write the measures (1.3) in the unified way:

(1.8) $$\Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0, 1, \\ K(Q||P), & s = 0, \\ K(P||Q), & s = 1. \end{cases}$$

Generalized Relative Information and Information Inequalities

Inder Jeet Taneja

Summarizing, we have the following particular cases of the measures (1.8):

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.

(ii) $\Phi_0(P||Q) = K(Q||P)$.

(iii) $\Phi_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$.

(iv) $\Phi_1(P||Q) = K(P||Q)$.

(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

2. Csiszár's $f-$Divergence and Information Bounds

Given a convex function $f : [0, \infty) \to \mathbb{R}$, the $f-$divergence measure introduced by Csiszár [4] is given by

$$C_f(p, q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{2.1}$$

where $p, q \in \mathbb{R}^n_+$.

The following two theorems can be seen in Csiszár and Körner [5].

Theorem 2.1. *(Joint convexity). If $f : [0, \infty) \to \mathbb{R}$ be convex, then $C_f(p, q)$ is jointly convex in p and q, where $p, q \in \mathbb{R}^n_+$.*

Theorem 2.2. *(Jensen's inequality). Let $f : [0, \infty) \to \mathbb{R}$ be a convex function. Then for any $p, q \in \mathbb{R}^n_+$, with $P_n = \sum_{i=1}^{n} p_i > 0, Q_n = \sum_{i=1}^{n} p_i > 0$, we have the inequality*

$$C_f(p, q) \geq Q_n f\left(\frac{P_n}{Q_n}\right).$$

The equality sign holds for strictly convex functions iff

$$\frac{p_1}{q_i} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}.$$

In particular, for all $P, Q \in \Delta_n$, we have

$$C_f(P||Q) \geq f(1),$$

with equality iff $P = Q$.

In view of Theorems 2.1 and 2.2, we have the following result.

Result 1. *For all $P, Q \in \Delta_n$, we have*

(i) $\Phi_s(P||Q) \geq 0$ *for any* $s \in \mathbb{R}$, *with equality iff* $P = Q$.

(ii) $\Phi_s(P||Q)$ *is convex function of the pair of distributions* $(P, Q) \in \Delta_n \times \Delta_n$ *and for any* $s \in \mathbb{R}$.

Proof. Take

$$\phi_s(u) = \begin{cases} [s(s-1)]^{-1}[u^s - 1 - s(u-1)], & s \neq 0, 1; \\ u - 1 - \ln u, & s = 0; \\ 1 - u + u \ln u, & s = 1 \end{cases} \tag{2.2}$$

for all $u > 0$ in (2.1), we have

$$C_f(P||Q) = \Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0, 1; \\ K(Q||P), & s = 0; \\ K(P||Q), & s = 1. \end{cases}$$

Moreover,

$$\phi'_s(u) = \begin{cases} (s-1)^{-1}(u^{s-1} - 1), & s \neq 0, 1; \\ 1 - u^{-1}, & s = 0; \\ \ln u, & s = 1 \end{cases} \tag{2.3}$$

and

$$\phi_s''(u) = \begin{cases} u^{s-2}, & s \neq 0, 1; \\ u^{-2}, & s = 0; \\ u^{-1}, & s = 1. \end{cases} \tag{2.4}$$

Thus we have $\phi_s''(u) > 0$ for all $u > 0$, and hence, $\phi_s(u)$ is strictly convex for all $u > 0$. Also, we have $\phi_s(1) = 0$. In view of Theorems 2.1 and 2.2 we have the proof of parts (i) and (ii) respectively. □

For some studies on the measure (2.2) refer to Liese and Vajda [15], Österreicher [17] and Cerone et al. [3].

The following theorem summarizes some of the results studies by Dragomir [7], [8]. For simplicity we have taken $f(1) = 0$ and $P, Q \in \Delta_n$.

Theorem 2.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex and normalized i.e., $f(1) = 0$. If $P, Q \in \Delta_n$ are such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

for some r and R with $0 < r \leq 1 \leq R < \infty$, then we have the following inequalities:

$$0 \leq C_f(P||Q) \leq \frac{1}{4}(R - r)\left(f'(R) - f'(r)\right), \tag{2.5}$$

$$0 \leq C_f(P||Q) \leq \beta_f(r, R), \tag{2.6}$$

and

$$0 \leq \beta_f(r,R) - C_f(P||Q) \leq \frac{f'(R)-f'(r)}{R-r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \leq \frac{1}{4}(R-r)\left(f'(R)-f'(r)\right), \tag{2.7}$$

where

$$\beta_f(r,R) = \frac{(R-1)f(r) + (1-r)f(R)}{R-r}, \tag{2.8}$$

and $\chi^2(P||Q)$ *and* $C_f(P||Q)$ *are as given by (1.6) and (2.1) respectively.*

In view of above theorem, we have the following result.

Result 2. *Let* $P, Q \in \Delta_n$ *and* $s \in \mathbb{R}$. *If there exists* r, R *such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

with $0 < r \leq 1 \leq R < \infty$, *then we have*

$$0 \leq \Phi_s(P||Q) \leq \mu_s(r,R), \tag{2.9}$$

$$0 \leq \Phi_s(P||Q) \leq \phi_s(r,R), \tag{2.10}$$

Generalized Relative Information and Information Inequalities

Inder Jeet Taneja

Title Page

Contents

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. 5(1) Art. 21, 2004
http://jipam.vu.edu.au

and

$$0 \le \phi_s(r,R) - \Phi_s(P||Q) \le k_s(r,R)\left[(R-1)(1-r) - \chi^2(P||Q)\right] \le \mu_s(r,R), \tag{2.11}$$

where

$$\mu_s(r,R) = \begin{cases} \frac{1}{4}\frac{(R-r)\left(R^{s-1}-r^{s-1}\right)}{(s-1)}, & s \ne 1; \\ \frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right), & s = 1 \end{cases} \tag{2.12}$$

$$\phi_s(r,R) = \frac{(R-1)\phi_s(r) + (1-r)\phi_s(R)}{R-r} = \begin{cases} \frac{(R-1)(r^s-1)+(1-r)(R^s-1)}{(R-r)s(s-1)}, & s \ne 0, 1; \\ \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{(R-r)}, & s = 0; \\ \frac{(R-1)r\ln r+(1-r)R\ln R}{(R-r)}, & s = 1, \end{cases} \tag{2.13}$$

and

$$k_s(r,R) = \frac{\phi_s'(R) - \phi_s'(r)}{R-r} = \begin{cases} \frac{R^{s-1}-r^{s-1}}{(R-r)(s-1)}, & s \ne 1; \\ \frac{\ln R - \ln r}{R-r}, & s = 1. \end{cases} \tag{2.14}$$

Proof. The above result follows immediately from Theorem 2.3, by taking $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then in this case we have $C_f(P||Q) = \Phi_s(P||Q)$. □

Moreover,

$$\mu_s(r,R) = \frac{1}{4}(R-r)^2\, k_s(r,R),$$

where

$$k_s(r,R) = \begin{cases} [L_{s-2}(r,R)]^{s-2}, & s \neq 1; \\ [L_{-1}(r,R)]^{-1} & s = 1, \end{cases}$$

and $L_p(a,b)$ is the famous (ref. Bullen, Mitrinović and Vasić [2]) *p-logarithmic mean* given by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0; \\ \frac{b-a}{\ln b - \ln a}, & p = -1; \\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p = 0, \end{cases}$$

for all $p \in \mathbb{R}$, $a, b \in \mathbb{R}_+$, $a \neq b$.

We have the following corollaries as particular cases of Result 2.

Corollary 2.4. *Under the conditions of Result 2, we have*

$$0 \leq \chi^2(Q||P) \leq \frac{1}{4}(R+r)\left(\frac{R-r}{rR}\right)^2, \tag{2.15}$$

$$0 \leq K(Q||P) \leq \frac{(R-r)^2}{4Rr}, \tag{2.16}$$

$$0 \leq K(P||Q) \leq \frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right), \tag{2.17}$$

$$0 \leq h(P||Q) \leq \frac{(R-r)\left(\sqrt{R}-\sqrt{r}\right)}{8\sqrt{Rr}} \tag{2.18}$$

and

$$0 \leq \chi^2(P||Q) \leq \frac{1}{2}(R-r)^2. \tag{2.19}$$

Proof. (2.15) follows by taking $s=-1$, (2.16) follows by taking $s=0$, (2.17) follows by taking $s=1$, (2.18) follows by taking $s=\frac{1}{2}$ and (2.19) follows by taking $s=2$ in (2.9). □

Corollary 2.5. *Under the conditions of Result 2, we have*

$$0 \leq \chi^2(Q||P) \leq \frac{(R-1)(1-r)}{rR}, \tag{2.20}$$

$$0 \leq K(Q||P) \leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}, \tag{2.21}$$

$$0 \leq K(P||Q) \leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}, \tag{2.22}$$

$$0 \leq h(P||Q) \leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} \tag{2.23}$$

and

$$0 \leq \chi^2(P||Q) \leq (R-1)(1-r). \tag{2.24}$$

Proof. (2.20) follows by taking $s = -1$, (2.21) follows by taking $s = 0$, (2.22) follows by taking $s = 1$, (2.23) follows by taking $s = \frac{1}{2}$ and (2.24) follows by taking $s = 2$ in (2.10). □

In view of (2.16), (2.17), (2.21) and (2.22), we have the following bounds on *J-divergence*:

$$0 \leq J(P||Q) \leq \min\{t_1(r,R), t_2(r,R)\}, \tag{2.25}$$

where

$$t_1(r,R) = \frac{1}{4}(R-r)^2\left[(rR)^{-1} + (L_{-1}(r,R))^{-1}\right]$$

and

$$t_2(r,R) = (R-1)(1-r)\left(L_{-1}(r,R)\right)^{-1}.$$

The expression $t_1(r,R)$ is due to (2.16) and (2.17) and the expression $t_2(r,R)$ is due to (2.21) and (2.22).

Corollary 2.6. *Under the conditions of Result 2, we have*

$$\begin{aligned} 0 &\leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ &\leq \frac{R+r}{(rR)^2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 &\leq \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\leq \frac{1}{rR}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{2.27}$$

Generalized Relative Information and Information Inequalities

Inder Jeet Taneja

Title Page

Contents

Go Back

Close

Quit

$$0 \leq \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \leq \frac{\ln R - \ln r}{R-r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \tag{2.28}$$

and

$$0 \leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\left(\sqrt{R}+\sqrt{r}\right)} - h(P||Q) \leq \frac{1}{2\sqrt{rR}\left(\sqrt{R}+\sqrt{r}\right)}\left[(R-1)(1-r) - \chi^2(P||Q)\right]. \tag{2.29}$$

Proof. (2.26) follows by taking $s = -1$, (2.27) follows by taking $s = 0$, (2.28) follows by taking $s = 1$, (2.29) follows by taking $s = \frac{1}{2}$ in (2.11). □

3. Main Results

In this section, we shall present a theorem generalizing the one obtained by Dragomir [9]. The results due to Dragomir [9] are limited only to χ^2- *divergence*, while the theorem established here is given in terms of *relative information of type* s, that in particular lead us to bounds in terms of χ^2-*divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination*.

Theorem 3.1. *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *the generating mapping be normalized, i.e.,* $f(1) = 0$ *and satisfy the assumptions:*

(i) f *is twice differentiable on* (r, R)*, where* $0 \leq r \leq 1 \leq R \leq \infty$;

(ii) there exists the real constants m, M *with* $m < M$ *such that*

$$m \leq x^{2-s} f''(x) \leq M, \quad \forall x \in (r, R), \quad s \in \mathbb{R}. \tag{3.1}$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} m\left[\phi_s(r,R) - \Phi_s(P||Q)\right] &\leq \beta_f(r,R) - C_f(P||Q) \\ &\leq M\left[\phi_s(r,R) - \Phi_s(P||Q)\right], \end{aligned} \tag{3.2}$$

where $C_f(P||Q), \Phi_s(P||Q), \beta_f(r,R)$ *and* $\phi_s(r,R)$ *are as given by (2.1), (1.8), (2.8) and (2.13) respectively.*

Proof. Let us consider the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ given by

$$F_{m,s}(u) = f(u) - m\phi_s(u), \tag{3.3}$$

and

$$F_{M,s}(u) = M\phi_s(u) - f(u), \tag{3.4}$$

respectively, where m and M are as given by (3.1) and function $\phi_s(\cdot)$ is as given by (2.3).

Since $f(u)$ and $\phi_s(u)$ are normalized, then $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are also normalized, i.e., $F_{m,s}(1) = 0$ and $F_{M,s}(1) = 0$. Moreover, the functions $f(u)$ and $\phi_s(u)$ are twice differentiable. Then in view of (2.4) and (3.1), we have

$$F''_{m,s}(u) = f''(u) - mu^{s-2} = u^{s-2}\left(u^{2-s}f''(u) - m\right) \geq 0$$

and

$$F''_{M,s}(u) = Mu^{s-2} - f''(u) = u^{s-2}\left(M - u^{2-s}f''(u)\right) \geq 0,$$

for all $u \in (r, R)$ and $s \in \mathbb{R}$. Thus the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are convex on (r, R).

We have seen above that the real mappings $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ defined over $\mathbb{R}_+$ given by (3.3) and (3.4) respectively are normalized, twice differentiable and convex on (r, R). Applying the r.h.s. of the inequality (2.6), we have

$$C_{F_{m,s}}(P||Q) \leq \beta_{F_{m,s}}(r, R), \tag{3.5}$$

and

$$C_{F_{m,s}}(P||Q) \leq \beta_{F_{M,s}}(r, R), \tag{3.6}$$

respectively.

Moreover,

$$C_{F_{m,s}}(P||Q) = C_f(P||Q) - m\Phi_s(P||Q), \tag{3.7}$$

and

$$C_{F_{M,s}}(P||Q) = M\Phi_s(P||Q) - C_f(P||Q). \tag{3.8}$$

In view of (3.5) and (3.7), we have

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_{F_{m,s}}(r, R),$$

i.e.,

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_f(r, R) - m\phi_s(r, R)$$

i.e.,

$$m\left[\phi_s(r, R) - \Phi_s(P||Q)\right] \leq \beta_f(r, R) - C_f(P||Q).$$

Thus, we have the l.h.s. of the inequality (3.2).

Again in view of (3.6) and (3.8), we have

$$M\Phi_s(P||Q) - C_f(P||Q) \leq \beta_{F_{M,s}}(r, R),$$

i.e.,

$$M\Phi_s(P||Q) - C_f(P||Q) \leq M\phi_s(r, R) - \beta_f(r, R),$$

i.e.,

$$\beta_f(r, R) - C_f(P||Q) \leq M\left[\phi_s(r, R) - \Phi_s(P||Q)\right].$$

Thus we have the r.h.s. of the inequality (3.2). □

Remark 3.1. *For similar kinds of results in comparing the $f-$divergence with Kullback-Leibler relative information see the work by Dragomir [10]. The case of Hellinger discrimination is discussed in Dragomir [6].*

We shall now present some particular case of the Theorem 3.1.

3.1. Information Bounds in Terms of χ^2-Divergence

In particular for $s = 2$, in Theorem 3.1, we have the following proposition:

Proposition 3.2. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) f is twice differentiable on (r, R), where $0 < r \le 1 \le R < \infty$;

(ii) there exists the real constants m, M with $m < M$ such that

$$m \le f''(x) \le M, \quad \forall x \in (r, R). \tag{3.9}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty,$$

then we have the inequalities:

$$\begin{aligned}\frac{m}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ \le \beta_f(r, R) - C_f(P||Q) \\ \le \frac{M}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right],\end{aligned} \tag{3.10}$$

where $C_f(P||Q), \beta_f(r,R)$ and $\chi^2(P||Q)$ are as given by (2.1), (2.8) and (1.6) respectively.

The above proposition was obtained by Dragomir in [9]. As a consequence of the above Proposition 3.2, we have the following result.

Result 3. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

then in view of Proposition 3.2, we have

$$\begin{aligned} \frac{R^{s-2}}{2}&\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \phi_s(r,R) - \Phi_s(P||Q) \\ &\leq \frac{r^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \quad s \leq 2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{r^{s-2}}{2}&\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \phi_s(r,R) - \Phi_s(P||Q) \\ &\leq \frac{R^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \quad s \geq 2. \end{aligned} \tag{3.12}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Now if $u \in [r, R] \subset (0, \infty)$, then we have

$$R^{s-2} \leq \phi_s''(u) \leq r^{s-2}, \quad s \leq 2,$$

or accordingly, we have

$$\phi_s''(u) \begin{cases} \leq r^{s-2}, & s \leq 2; \\ \geq r^{s-2}, & s \geq 2 \end{cases} \tag{3.13}$$

and

$$\phi_s''(u) \begin{cases} \leq R^{s-2}, & s \geq 2; \\ \geq R^{s-2}, & s \leq 2, \end{cases} \tag{3.14}$$

where r and R are as defined above. Thus in view of (3.9), (3.13) and (3.14), we have the proof. □

In view of Result 3, we have the following corollary.

Corollary 3.3. *Under the conditions of Result 3, we have*

$$\begin{aligned} &\frac{1}{R^3}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\quad \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ &\quad \leq \frac{1}{r^3}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{3.15}$$

Generalized Relative Information and Information Inequalities

Inder Jeet Taneja

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

$$(3.16)\quad \frac{1}{2R^2}\left[(R-1)(1-r)-\chi^2(P||Q)\right] \leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P) \leq \frac{1}{2r^2}\left[(R-1)(1-r)-\chi^2(P||Q)\right],$$

$$(3.17)\quad \frac{1}{2R}\left[(R-1)(1-r)-\chi^2(P||Q)\right] \leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q) \leq \frac{1}{2r}\left[(R-1)(1-r)-\chi^2(P||Q)\right]$$

and

$$(3.18)\quad \frac{1}{8\sqrt{R^3}}\left[(R-1)(1-r)-\chi^2(P||Q)\right] \leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q) \leq \frac{1}{8\sqrt{r^3}}\left[(R-1)(1-r)-\chi^2(P||Q)\right].$$

Proof. (3.15) follows by taking $s=-1$, (3.16) follows by taking $s=0$, (3.17) follows by taking $s=1$, (3.18) follows by taking $s=\frac{1}{2}$ in Result 3. While for $s=2$, we have equality sign. □

Proposition 3.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) f is twice differentiable on (r, R), where $0 < r \leq 1 \leq R < \infty$;

(ii) there exists the real constants m, M such that $m < M$ and

$$m \leq x^3 f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.19}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} \frac{m}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] &\\ \leq \beta_f(r, R) - C_f(P||Q) &\\ \leq \frac{M}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], & \end{aligned} \tag{3.20}$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $\chi^2(Q||P)$ are as given by (2.1), (2.8) and (1.7) respectively.

As a consequence of above proposition, we have the following result.

Result 4. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.4, we have

$$(3.21)\qquad \frac{R^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right] \leq \phi_s(r,R)-\Phi_s(P||Q) \leq \frac{r^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right], \quad s\leq -1$$

and

$$(3.22)\qquad \frac{r^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right] \leq \phi_s(r,R)-\Phi_s(P||Q) \leq \frac{R^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right], \quad s\geq -1.$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r,R] \to \mathbb{R}$ such that $g(u) = u^3\phi_s''(u) = u^{s+1}$, then we have

$$(3.23)\qquad \sup_{u\in[r,R]} g(u) = \begin{cases} R^{s+1}, & s\geq -1; \\ r^{s+1}, & s\leq -1 \end{cases}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s+1}, & s \geq -1; \\ R^{s+1}, & s \leq -1. \end{cases} \tag{3.24}$$

In view of (3.23) , (3.24) and Proposition 3.4, we have the proof of the result.

□

In view of Result 4, we have the following corollary.

Corollary 3.5. *Under the conditions of Result 4, we have*

$$\begin{aligned} &\frac{r}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\leq \frac{R}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\frac{r^2}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \\ &\leq \frac{R^2}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \end{aligned} \tag{3.26}$$

(3.27)
$$\frac{\sqrt{r^3}}{8}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right] \leq \frac{\left(\sqrt{R}-1\right)\left(1-\sqrt{r}\right)}{\sqrt{R}+\sqrt{r}} - h(P||Q) \leq \frac{\sqrt{R^3}}{8}\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right]$$

and

(3.28)
$$r^3\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right] \leq (R-1)(1-r)-\chi^2(P||Q) \leq R^3\left[\frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)\right].$$

Proof. (3.25) follows by taking $s=0$, (3.26) follows by taking $s=1$, (3.27) follows by taking $s=\frac{1}{2}$ and (3.28) follows by taking $s=2$ in Result 4. While for $s=-1$, we have equality sign. □

3.2. Information Bounds in Terms of Kullback-Leibler Relative Information

In particular for $s=1$, in the Theorem 3.1, we have the following proposition (see also Dragomir [10]).

Proposition 3.6. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 < r \le 1 \le R < \infty$;*

(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \le x f''(x) \le M, \quad \forall x \in (r, R). \tag{3.29}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty,$$

then we have the inequalities:

$$\begin{aligned} m &\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ &\le \beta_f(r, R) - C_f(P||Q) \\ &\le M\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \end{aligned} \tag{3.30}$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $K(P||Q)$ as given by (2.1), (2.8) and (1.1) respectively.

In view of the above proposition, we have the following result.

Result 5. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \le 1 \le R < \infty)$ such that*

$$0 < r \le \frac{p_i}{q_i} \le R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.6, we have

$$(3.31)\quad r^{s-1}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \le \phi_s(r,R)-\Phi_s(P||Q) \le R^{s-1}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right],\quad s\ge 1$$

and

$$(3.32)\quad R^{s-1}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \le \phi_s(r,R)-\Phi_s(P||Q) \le r^{s-1}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right],\quad s\le 1.$$

Proof. Let us consider $f(u)=\phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u)=u^{s-2}.$$

Let us define the function $g:[r,R]\to\mathbb{R}$ such that $g(u)=\phi_s''(u)=u^{s-1}$, then we have

$$(3.33)\quad \sup_{u\in[r,R]} g(u)=\begin{cases}R^{s-1}, & s\ge 1;\\ r^{s-1}, & s\le 1\end{cases}$$

Generalized Relative Information and Information Inequalities

Inder Jeet Taneja

Title Page

Contents

Go Back

Close

Quit

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s-1}, & s \geq 1; \\ R^{s-1}, & s \leq 1. \end{cases} \tag{3.34}$$

In view of (3.33), (3.34) and Proposition 3.6 we have the proof of the result.

□

In view of Result 5, we have the following corollary.

Corollary 3.7. *Under the conditions of Result 5, we have*

$$\begin{aligned} \frac{2}{R^2}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q)\right] &\\ \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) &\\ \leq \frac{2}{r^2}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q)\right], & \end{aligned} \tag{3.35}$$

$$\begin{aligned} \frac{1}{R}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q)\right] &\\ \leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) &\\ \leq \frac{1}{r}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q)\right], & \end{aligned} \tag{3.36}$$

$$\frac{1}{4\sqrt{R}}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \tag{3.37}$$
$$\leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)$$
$$\leq \frac{1}{4\sqrt{r}}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right]$$

and

$$2r\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \tag{3.38}$$
$$\leq (R-1)(1-r)-\chi^2(P||Q)$$
$$\leq 2R\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right].$$

Proof. (3.35) follows by taking $s=-1$, (3.36) follows by taking $s=0$, (3.37) follows by taking $s=\frac{1}{2}$ and (3.38) follows by taking $s=2$ in Result 5. For $s=1$, we have equality sign. □

In particular, for $s=0$ in Theorem 3.1, we have the following proposition:

Proposition 3.8. *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ the generating mapping be normalized, i.e., $f(1)=0$ and satisfy the assumptions:*

(i) f is twice differentiable on (r,R), where $0<r\leq 1\leq R<\infty$;

(ii) *there exists the real constants* m, M *with* $m < M$ *such that*

(3.39) $$m \leq x^2 f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

(3.40) $$m \left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \right] \leq \beta_f(r, R) - C_f(P||Q) \leq M \left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \right],$$

where $C_f(P||Q), \beta_f(r, R)$ *and* $K(Q||P)$ *as given by (2.1), (2.8) and (1.1) respectively.*

In view of Proposition 3.8, we have the following result.

Result 6. *Let* $P, Q \in \Delta_n$ *and* $s \in \mathbb{R}$. *Let there exist* r, R $(0 < r \leq 1 \leq R < \infty)$ *such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.8, we have

$$r^s\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \le \phi_s(r,R)-\Phi_s(P||Q) \le R^s\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right],\quad s\ge 0 \tag{3.41}$$

and

$$R^s\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \le \phi_s(r,R)-\Phi_s(P||Q) \le r^s\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right],\quad s\le 0. \tag{3.42}$$

Proof. Let us consider $f(u)=\phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u)=u^{s-2}.$$

Let us define the function $g:[r,R]\to\mathbb{R}$ such that $g(u)=u^2\phi_s''(u)=u^s$, then we have

$$\sup_{u\in[r,R]} g(u)=\begin{cases} R^s, & s\ge 0;\\ r^s, & s\le 0\end{cases} \tag{3.43}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^s, & s \geq 0; \\ R^s, & s \leq 0. \end{cases} \tag{3.44}$$

In view of (3.43), (3.44) and Proposition 3.8, we have the proof of the result. □

In view of Result 6, we have the following corollary.

Corollary 3.9. *Under the conditions of Result 6, we have*

$$\begin{aligned} \frac{2}{R}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] &\leq \frac{(R-1)(1-r)}{rR}-\chi^2(Q||P) \\ &\leq \frac{2}{r}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right], \end{aligned} \tag{3.45}$$

$$\begin{aligned} r\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] &\leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q) \\ &\leq R\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right], \end{aligned} \tag{3.46}$$

$$\frac{\sqrt{r}}{4}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right]$$
$$\leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)$$
$$\leq \frac{\sqrt{R}}{4}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \tag{3.47}$$

and

$$2r^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right]$$
$$\leq (R-1)(1-r)-\chi^2(P||Q)$$
$$\leq 2R^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right]. \tag{3.48}$$

Proof. (3.45) follows by taking $s=-1$, (3.46) follows by taking $s=1$, (3.47) follows by taking $s=\frac{1}{2}$ and (3.48) follows by taking $s=2$ in Result 6. For $s=0$, we have equality sign. □

3.3. Information Bounds in Terms of Hellinger's Discrimination

In particular, for $s=\frac{1}{2}$ in Theorem 3.1, we have the following proposition (see also Dragomir [6]).

Proposition 3.10. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) f is twice differentiable on (r, R), where $0 < r \leq 1 \leq R < \infty$;

(ii) there exists the real constants m, M with $m < M$ such that

$$m \leq x^{3/2} f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.49}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} 4m \left[\frac{\left(\sqrt{R} - 1\right)\left(1 - \sqrt{r}\right)}{\sqrt{R} + \sqrt{r}} - h(P||Q) \right] &\leq \beta_f(r, R) - C_f(P||Q) \\ &\leq 4M \left[\frac{\left(\sqrt{R} - 1\right)\left(1 - \sqrt{r}\right)}{\sqrt{R} + \sqrt{r}} - h(P||Q) \right], \end{aligned} \tag{3.50}$$

where $C_f(P||Q), \beta_f(r, R)$ and $h(P||Q)$ as given by (2.1), (2.8) and (1.5) respectively.

In view of Proposition 3.10, we have the following result.

Result 7. *Let* $P, Q \in \Delta_n$ *and* $s \in \mathbb{R}$. *Let there exist* r, R $(0 < r \leq 1 \leq R < \infty)$ *such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.10, we have

$$\begin{aligned} (3.51) \qquad & 4r^{\frac{2s-1}{2}} \left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq 4R^{\frac{2s-1}{2}} \left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right], \quad s \geq \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} (3.52) \qquad & 4R^{\frac{2s-1}{2}} \left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq 4r^{\frac{2s-1}{2}} \left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right], \quad s \leq \frac{1}{2}. \end{aligned}$$

Proof. Let the function $\phi_s(u)$ given by (3.29) be defined over $[r, R]$. Defining

$g(u) = u^{3/2}\phi_s''(u) = u^{\frac{2s-1}{2}}$, obviously we have

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ r^{\frac{2s-1}{2}}, & s \leq \frac{1}{2} \end{cases} \tag{3.53}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ R^{\frac{2s-1}{2}}, & s \leq \frac{1}{2}. \end{cases} \tag{3.54}$$

In view of (3.53), (3.54) and Proposition 3.10, we get the proof of the result. □

In view of Result 7, we have the following corollary.

Corollary 3.11. *Under the conditions of Result 7, we have*

$$\begin{aligned} \frac{8}{\sqrt{R^3}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q)\right] &\leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ &\leq \frac{8}{\sqrt{r^3}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q)\right], \end{aligned} \tag{3.55}$$

$$(3.56)\quad \frac{4}{\sqrt{R}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]$$
$$\leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)$$
$$\leq \frac{4}{\sqrt{r}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],$$

$$(3.57)\quad 4\sqrt{r}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]$$
$$\leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)$$
$$\leq 4\sqrt{R}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]$$

and

$$(3.58)\quad 8\sqrt{r^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]$$
$$\leq (R-1)(1-r)-\chi^2(P||Q)$$

$$\leq 8\sqrt{R^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q)\right].$$

Proof. (3.55) follows by taking $s=-1$, (3.56) follows by taking $s=0$, (3.57) follows by taking $s=1$ and (3.58) follows by taking $s=2$ in Result 7. For $s=\frac{1}{2}$, we have equality sign. □

References

[1] A. BHATTACHARYYA, Some analogues to the amount of information and their uses in statistical estimation, *Sankhya*, **8** (1946), 1–14.

[2] P.S. BULLEN, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Means and Their Inequalities*, Kluwer Academic Publishers, 1988.

[3] P. CERONE, S.S. DRAGOMIR AND F. ÖSTERREICHER, Bounds on extended f-divergences for a variety of classes, *RGMIA Research Report Collection*, **6**(1) (2003), Article 7.

[4] I. CSISZÁR, Information type measures of differences of probability distribution and indirect observations, *Studia Math. Hungarica*, **2** (1967), 299–318.

[5] I. CSISZÁR AND J. KÖRNER, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1981.

[6] S.S. DRAGOMIR, Upper and lower bounds for Csiszár $f-$divergence in terms of Hellinger discrimination and applications, *Nonlinear Analysis Forum*, **7**(1) (2002), 1–13

[7] S.S. DRAGOMIR, Some inequalities for the Csiszár Φ-divergence - *Inequalities for Csiszár f-Divergence in Information Theory*, http://rgmia.vu.edu.au/monographs/csiszar.htm

[8] S.S. DRAGOMIR, A converse inequality for the Csiszár Φ-Divergence- *Inequalities for Csiszár f-Divergence in Information Theory*, http://rgmia.vu.edu.au/monographs/csiszar.htm

[9] S.S. DRAGOMIR, Other inequalities for Csiszár divergence and applications - *Inequalities for Csiszár f-Divergence in Information Theory*, http://rgmia.vu.edu.au/monographs/csiszar.htm

[10] S.S. DRAGOMIR, Upper and lower bounds Csiszár $f-$divergence in terms of Kullback-Leibler distance and applications - *Inequalities for Csiszár f-Divergence in Information Theory*, http://rgmia.vu.edu.au/monographs/csiszar.htm

[11] E. HELLINGER, Neue Begründung der Theorie der quadratischen Formen von unendlichen vielen Veränderlichen, *J. Reine Aug. Math.*, **136** (1909), 210–271.

[12] H. JEFFREYS, An invariant form for the prior probability in estimation problems, *Proc. Roy. Soc. Lon., Ser. A*, **186** (1946), 453–461.

[13] S. KULLBACK AND R.A. LEIBLER, On information and sufficiency, *Ann. Math. Statist.*, **22** (1951), 79–86.

[14] L. LECAM, *Asymptotic Methods in Statistical Decision Theory*, New York, Springer, 1978.

[15] F. LIESE AND I. VAJDA, *Convex Statistical Decision Rules*, Teubner-Texte zur Mathematick, Band 95, Leipzig, 1987.

[16] K. PEARSON, On the criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling, *Phil. Mag.*, **50** (1900), 157–172.

[17] F. ÖSTERREICHER, Csiszár's f-Divergence – Basic Properties – pre-print, 2002. http://rgmia.vu.edu.au

[18] A. RÉNYI, On measures of entropy and information, *Proc. 4th Berk. Symp. Math. Statist. and Probl.*, University of California Press, **Vol. 1** (1961), 547–461.

[19] R. SIBSON, Information radius, *Z. Wahrs. und verw Geb.*, **14** (1969), 149–160.

[20] I.J. TANEJA, New developments in generalized information measures, Chapter in: *Advances in Imaging and Electron Physics*, Ed. P.W. Hawkes, **91**(1995), 37-135.

[21] I.J. TANEJA, *Generalized Information Measures and their Applications*, 2001, [ONLINE http://www.mtm.ufsc.br/~taneja/book/book.html.]

[22] I. VAJDA, *Theory of Statistical Inference and Information*, Kluwer Academic Press, London, 1989.