Journal of Inequalities in Pure and Applied Mathematics
http://jipam.vu.edu.au/
Volume 6, Issue 2, Article 41, 2005

SOME APPLICATIONS OF THE BRIOT-BOUQUET DIFFERENTIAL SUBORDINATION

H.M. SRIVASTAVA AND A.Y. LASHIN
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA V8W 3P4, CANADA
harimsri@math.uvic.ca

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE, MANSOURA UNIVERSITY
MANSOURA 35516, EGYPT
aylashin@yahoo.com

Received 14 March, 2005; accepted 19 March, 2005
Communicated by Th.M. Rassias

ABSTRACT. By using a method based upon the Briot-Bouquet differential subordination, we prove several subordination results involving starlike and convex functions of complex order. Some special cases and consequences of the main subordination results are also indicated.

Key words and phrases: Analytic functions, Univalent functions, Starlike functions of complex order, Convex functions of complex order, Differential subordinations, Schwarz function.

2000 *Mathematics Subject Classification.* Primary 26D07, 30C45; Secondary 26D20.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *starlike of complex order* b $(b \in \mathbb{C} \setminus \{0\})$ in $\mathbb{U}$ if and only if

$$\frac{f(z)}{z} \neq 0 \text{ and } \Re\left(1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0 \qquad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \{0\}). \tag{1.2}$$

ISSN (electronic): 1443-5756

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.
078-05

We denote by $\mathcal{S}_0^*(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order b in $\mathbb{U}$. Further, let $\mathcal{S}_1^*(b)$ denote the class of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < |b| \quad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \{0\}). \tag{1.3}$$

We note that $\mathcal{S}_1^*(b)$ is a subclass of $\mathcal{S}_0^*(b)$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *convex of complex order* b $(b \in \mathbb{C}\setminus\{0\})$ in $\mathbb{U}$ if and only if

$$\frac{f(z)}{z} \neq 0 \quad \text{and} \quad \Re\left(1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \{0\}). \tag{1.4}$$

We denote by $\mathcal{K}_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order b in $\mathbb{U}$. Furthermore, let $\mathcal{K}_1(b)$ denote the class of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$\left|\frac{zf''(z)}{f'(z)}\right| < |b| \quad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \{0\}), \tag{1.5}$$

so that, obviously, $\mathcal{K}_1^*(b)$ is a subclass of $\mathcal{K}_0^*(b)$.

We note that

$$f(z) \in \mathcal{K}_0(b) \Leftrightarrow zf^{'}(z) \in \mathcal{S}_0^*(b) \qquad (b \in \mathbb{C} \setminus \{0\}) \tag{1.6}$$

and

$$f(z) \in \mathcal{K}_1(b) \Leftrightarrow zf^{'}(z) \in \mathcal{S}_1^*(b) \qquad (b \in \mathbb{C} \setminus \{0\}). \tag{1.7}$$

The classes $\mathcal{S}_0^*(b)$ and $\mathcal{K}_0(b)$ of starlike and convex functions of a *complex* order b in $\mathbb{U}$ were introduced and investigated earlier by Nasr and Aouf [8] and Wiatrowski [12], respectively (see also [6], [7] and [9]). Their subclasses $\mathcal{S}_1^*(b)$ and $\mathcal{K}_1(b)$ were studied by (among others) Choi [1] (see also Choi and Saigo [2]), Polatoğlu and Bolcal [10] and Lashin [4].

Remark 1. *Upon setting* $b = 1 - \alpha$ $(0 \leqq \alpha < 1)$, *we observe that*

$$\mathcal{S}_0^*(1-\alpha) = \mathcal{S}^*(\alpha) \ \text{ and } \ \mathcal{K}_0(1-\alpha) = \mathcal{K}(\alpha),$$

where $\mathcal{S}^*(\alpha)$ *and* $\mathcal{K}(\alpha)$ *denote, respectively, the relatively more familiar classes of starlike and convex functions of a real order* α *in* $\mathbb{U}$ (*see, for example,* [11]).

Finally, for two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}). \tag{1.8}$$

In particular, if the function g is *univalent* in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.9}$$

The main object of the present sequel to the aforementioned works is to apply a method based upon the Briot-Bouquet differential subordination in order to derive several subordination results involving starlike and convex functions of complex order. We also indicate some interesting special cases and consequences of our main subordination results.

2. MAIN SUBORDINATION RESULTS

In order to prove our main subordination results, we shall make use of the following known results.

Lemma 1 (*cf.* Miller and Mocanu [5, p. 17 *et seq.*]). *Let the functions $F(z)$ and $G(z)$ be analytic in the open unit disk $\mathbb{U}$ and let*

$$F(0) = G(0).$$

If the function $H(z) := zG'(z)$ is starlike in $\mathbb{U}$ and

$$zF'(z) \prec zG'(z) \qquad (z \in \mathbb{U}),$$

then

$$F(z) \prec G(z) = G(0) + \int_0^z \frac{H(t)}{t}\,dt \qquad (z \in \mathbb{U}). \tag{2.1}$$

The function $G(z)$ is convex and is the best dominant in (2.1).

Lemma 2 (Eenigenburg *et al.* [3]). *Let β and γ be complex constants. Also let the function $h(z)$ be convex (univalent) in $\mathbb{U}$ with*

$$h(0) = 1 \quad \textit{and} \quad \Re\big(\beta h(z) + \gamma\big) > 0 \qquad (z \in \mathbb{U}).$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

is analytic in $\mathbb{U}$ and satisfies the following differential subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.2}$$

If the differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad \big(q(0) := 1\big) \tag{2.3}$$

has a univalent solution $q(z)$, then

$$p(z) \prec q(z) \prec h(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant in (2.2) $\big($that is, $p(z) \prec q(z)$ $(z \in \mathbb{U})$ for all $p(z)$ satisfying (2.2) and if $p(z) \prec \hat{q}(z)$ $(z \in \mathbb{U})$ for all $p(z)$ satisfying (2.2), then $q(z) \prec \hat{q}(z)\big)$ $(z \in \mathbb{U})$.

Remark 2. *The conclusion of Lemma 2 can be written in the following form:*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \;\Rightarrow\; p(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Remark 3. *The differential equation (2.3) has its formal solution given by*

$$q(z) = \frac{zF'(z)}{F(z)} = \frac{\beta + \gamma}{\beta}\left(\frac{H(z)}{F(z)}\right)^{\beta} - \frac{\gamma}{\beta},$$

where

$$F(z) = \left(\frac{\beta + \gamma}{z^{\gamma}} \int_0^z \{H(t)\}^{\beta} t^{\gamma - 1}\,dt\right)^{\frac{1}{\beta}}$$

and

$$H(z) = z \cdot \exp\left(\int_0^z \frac{h(t) - 1}{t}\,dt\right).$$

We now state our first subordination result given by Theorem 1 below.

Theorem 1. *Let the function* $h(z)$ *be convex* (*univalent*) *in* $\mathbb{U}$ *and let*

$$h(0) = 1 \quad \textit{and} \quad \Re\big(bh(z) + (1-b)\big) > 0 \qquad (z \in \mathbb{U}).$$

Also let $f(z) \in \mathcal{A}$.

(a) *If*

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec h(z) \qquad (z \in \mathbb{U}), \tag{2.4}$$

then

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.5}$$

(b) *If the following differential equation*:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad \big(q(0) := 1\big)$$

has a univalent solution $q(z)$, *then*

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec h(z) \Rightarrow 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec q(z) \prec h(z) \qquad (z \in \mathbb{U}) \tag{2.6}$$

and $q(z)$ *is the best dominant in* (2.6).

Proof. We begin by setting

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) =: p(z), \tag{2.7}$$

so that $p(z)$ has the following series expansion:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots .$$

By differentiating (2.7) logarithmically, we obtain

$$p(z) + \frac{zp'(z)}{bp(z) + (1-b)} = 1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}$$

and the subordination (2.4) can be written as follows:

$$p(z) + \frac{zp'(z)}{bp(z) + (1-b)} \prec h(z) \qquad (z \in \mathbb{U}).$$

Now the conclusions of the theorem would follow from Lemma 2 by taking

$$\beta = b \quad \text{and} \quad \gamma = 1 - b.$$

This evidently completes the proof of Theorem 1. $\square$

Next we prove Theorem 2 below.

Theorem 2. *If* $f(z) \in \mathcal{K}_1(b) \quad (|b| \leqq 1;\ b \neq 0)$, *then*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec q(z) \qquad (z \in \mathbb{U}),$$

where $q(z)$ *is the best dominant given by*

$$q(z) = 1 - \frac{1}{b} + \frac{ze^{bz}}{e^{bz} - 1}. \tag{2.8}$$

Proof. First of all, we observe that (1.5) is equivalent to the following inequality:

$$\left|\left(1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right)-1\right|<1 \qquad (z\in\mathbb{U}),$$

which implies that

$$1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\prec 1+z \qquad (z\in\mathbb{U}).$$

Thus, in Theorem 1, we choose

$$h(z)=1+z$$

and note that

$$\Re\big(bh(z)+(1-b)\big)>0 \text{ when } z\in\mathbb{U} \text{ and } |b|\leqq 1\,(b\neq 0),$$

and $h(z)$ satisfies the hypotheses of Lemma 2. Consequently, in the view of Lemma 2 and Remark 3, we have

$$H(z)=z\cdot\exp\left(\int_0^z \frac{h(t)-1}{t}\,dt\right)$$

which, for $h(t)=1+t$, yields

$$H(z)=ze^z \tag{2.9}$$

and

$$F(z)=\left(\frac{1}{z^{1-b}}\int_0^z\left[\frac{H(t)}{t}\right]^b dt\right)^{\frac{1}{b}},$$

that is,

$$F(z)=\left(\frac{1}{z^{1-b}}\int_0^z e^{bt}dt\right)^{\frac{1}{b}},$$

which readily simplifies to the following form:

$$F(z)=\left(\frac{1}{bz^{1-b}}\left(e^{bz}-1\right)\right)^{\frac{1}{b}}, \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$q(z)=\frac{1}{b}\left(\frac{H(z)}{F(z)}\right)^b-\frac{1-b}{b},$$

which leads us easily to (2.8), thereby completing our proof of Theorem 2. □

Lastly, we prove the following subordination result.

Theorem 3. *Let* $f(z)\in\mathcal{S}_0^*(b)$ $(b\in\mathbb{C}\setminus\{0\})$, *then*

$$\frac{f(z)}{z}\prec\frac{1}{(1-z)^{2b}} \qquad (z\in\mathbb{U}) \tag{2.11}$$

and this is the best dominant.

Proof. Since $f(z)\in\mathcal{S}_0^*(b)$ $(b\in\setminus\{0\})$, we have

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z} \qquad (z\in\mathbb{U}),$$

that is,

$$\frac{1}{b}\left(\frac{zf'(z)}{f(z)}+1\right)\prec\frac{2z}{1-z}+\frac{2}{b} \qquad (z\in\mathbb{U}). \tag{2.12}$$

Now, by setting

$$P(z) := \big(zf(z)\big)^{\frac{1}{b}} \qquad (z \in \mathbb{U}),$$

we can rewrite (2.12) in the following form:

$$z\big(\log P(z)\big)' \prec z\left(\log\left[z^{\frac{2}{b}}(1-z)^{-2}\right]\right)' \qquad (z \in \mathbb{U}).$$

Thus, by setting

$$F(z) = \log P(z) \quad \text{and} \quad G(z) = \log\left[z^{\frac{2}{b}}(1-z)^{-2}\right]$$

in Lemma 1, we find that

$$\log P(z) \prec \log\left[z^{\frac{2}{b}}(1-z)^{-2}\right] \qquad (z \in \mathbb{U}),$$

which obviously is equivalent to the assertion (2.11) of Theorem 3. □

3. Some Interesting Deductions

In view especially of the equivalence relationships exhibited by (1.6) and (1.7), each of our main results proven in the preceding section can indeed be applied to yield the corresponding subordination results involving convex functions of order $b \in \mathbb{C} \setminus \{0\}$. For example, Theorem 3 would immediately lead us to the following subordination result.

Corollary 1. *Let* $f(z) \in \mathcal{K}_0(b)$ $(b \in \mathbb{C} \setminus \{0\})$. *Then*

$$f'(z) \prec \frac{1}{(1-z)^{2b}} \qquad (z \in \mathbb{U})$$

and this is the best dominant.

References

[1] J.H. CHOI, Starlike and convex function of complex order involving a certain fractional integral operator, in *New Developments in Convolution* (Japanese) (Kyoto, 1997), *Sūrikaisekikenkyūsho Kōkyūroku*, **1012** (1997), 1–13.

[2] J.H. CHOI AND M. SAIGO, Starlike and convex functions involving a certain fractional integral operator, *Fukuoka Univ. Sci. Rep.*, **28**(2) (1998), 29–40.

[3] P. EENIGENBURG, S.S. MILLER, P.T. MOCANU AND M.O. READ, On a Briot-Bouquet differential subordination, in *General Inequalities* **3**, pp. 339–348, International Series of Numerical Mathematics, Vol. **64**, Birkhäuser Verlag, Basel, 1983; see also *Rev. Roumaine Math. Pures Appl.*, **29** (1984), 567–573.

[4] A.Y. LASHIN, Starlike and convex functions of complex order involving a certain linear operator, *Indian J. Pure Appl. Math.*, **34** (2003), 1101–1108.

[5] S.S. MILLER AND P.T. MOCANU, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics (No. 225), Marcel Dekker, New York and Basel, 2000.

[6] G. MURUGUSUNDARAMOORTHY AND H.M. SRIVASTAVA, Neighborhoods of certain classes of analytic functions of complex order, *J. Inequal. Pure Appl. Math.*, **5**(2) (2004), Art. 24, pp. 1-8. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=374`]

[7] M.A. NASR AND M.K. AOUF, On convex functions of complex order, *Mansoura Sci. Bull. Egypt*, **9** (1982), 565–582.

[8] M.A. NASR AND M.K. AOUF, Starlike function of complex order, *J. Natur. Sci. Math.*, **25** (1985), 1–12.

[9] S. OWA AND G.S. SĂLĂGEAN, On an open problem of S. Owa, in *New Developments in Convolution* (Japanese) (Kyoto, 1997), *Sūrikaisekikenkyūsho Kōkyūroku* **1012** (1997), 110–114; see also *J. Math. Anal. Appl.*, **218** (1998), 453–457.

[10] Y. POLATOĞLU AND M. BOLCAL, The radius of convexity for the class of Janowski convex functions of complex order, *Mat. Vesnik,* **54** (2002), 9–12.

[11] H.M. SRIVASTAVA AND S. OWA (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[12] P. WIATROWSKI, On the coefficients of some family of holomorphic functions, *Zeszyty Nauk. Uniw. Łódz Nauk. Mat.-Przyrod.* (*Ser.* 2), **39** (1970), 75–85.