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ON LANDAU TYPE INEQUALITIES FOR FUNCTIONS WITH HÖLDER CONTINUOUS DERIVATIVES

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ABSTRACT. An inequality of Landau type for functions whose derivatives satisfy Hölder's condition is studied.

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1. INTRODUCTION

S.S. Dragomir and C.I. Preda have proved the following theorem (see [1]):

Theorem A. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ locally absolutely continuous function on I. If $f \in L_\infty(I)$ and the derivative $f' : I \to \mathbb{R}$ satisfies Hölder's condition*

$$|f'(t) - f'(s)| \leq H \cdot |t-s|^\alpha \quad \textit{for any} \quad t, s \in I, \tag{1.1}$$

where $H > 0$ and $\alpha \in (0,1]$ are given, then $f' \in L_\infty(I)$ and one has the inequalities:

$$||f'|| \leq \begin{cases} \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} \\ \qquad \textit{if} \quad m(I) \geq 2^{\frac{\alpha+2}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}; \\ \frac{4\cdot||f||}{m(I)} + \frac{H}{2^\alpha(\alpha+1)}[m(I)]^\alpha \\ \qquad \textit{if} \quad 0 < m(I) \leq 2^{\frac{\alpha+2}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}, \end{cases} \tag{1.2}$$

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where $||\cdot||$ *is the* ∞*-norm on the interval* I*, and* $m(I)$ *is the length of* I*.*

In our paper we shall give an improvement of this theorem.

2. MAIN RESULTS

Theorem 2.1. *Let* I *be an interval and* $f : I \to \mathbb{R}$ *function on* I *satisfying conditions of Theorem A. Then* $f' \in L_\infty(I)$ *and the following inequlities hold:*

$$||f'|| \le \begin{cases} \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} \\ \qquad \text{if} \quad m(I) \ge 2^{\frac{1}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}; \\ \frac{2||f||}{m(I)} + \frac{H}{\alpha+1}\left[m(I)\right]^\alpha \\ \qquad \text{if} \quad 0 < m(I) \le 2^{\frac{1}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}, \end{cases} \tag{2.1}$$

where $||\cdot||$ *is the* ∞*-norm on the interval* I*, and* $m(I)$ *is the length of* I*.*

In our proof and in the subsequent discussion we use three lemmas.

Lemma 2.2. *Let* $a, b \in \mathbb{R}$*,* $a < b$*,* $\alpha \in (0, 1]$*. Then the following inequality holds:*

$$(b-x)^{\alpha+1} + (x-a)^{\alpha+1} \le (b-a)^{\alpha+1}, \quad \forall x \in [a,b]. \tag{2.2}$$

Proof. Consider the function $y : [a, b] \to \mathbb{R}$ given by:

$$y(x) = (b-x)^{\alpha+1} + (x-a)^{\alpha+1}.$$

We observe that the unique solution of the equation

$$y'(x) = (\alpha+1)\left[(x-a)^\alpha - (b-x)^\alpha\right] = 0$$

is $x_0 = \frac{a+b}{2} \in [a, b]$. The function $y'(x)$ is decreasing on (a, x_0) and increasing on (x_0, b). Thus, the maximal values for $y(x)$ are attained on the boundary of $[a, b] : y(a) = y(b) = (b-a)^{\alpha+1}$, which proves the lemma. □

A generalization of the following lemma is proved in [1]:

Lemma 2.3. *Let* $A, B > 0$ *and* $\alpha \in (0, 1]$*. Consider the function* $g_\alpha : (0, \infty) \to \mathbb{R}$ *given by:*

$$g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha. \tag{2.3}$$

Define $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0, \infty)$*. Then for* $\lambda_1 \in (0, \infty)$ *we have the bound*

$$\inf_{\lambda \in (0, \lambda_1]} g_\alpha(\lambda) = \begin{cases} \frac{A}{\lambda_1} + B \cdot \lambda_1^\alpha & \text{if} \quad 0 < \lambda_1 < \lambda_0 \\ (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}} & \text{if} \quad \lambda_1 \ge \lambda_0. \end{cases} \tag{2.4}$$

Proof. We have:

$$g'_\alpha(\lambda) = -\frac{A}{\lambda^2} + \alpha \cdot B \cdot \lambda^{\alpha-1}.$$

The unique solution of the equation $g'_\alpha(\lambda) = 0$, $\lambda \in (0, \infty)$, is $\lambda_0 = \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0, \infty)$. The function $g_\alpha(\lambda)$ is decreasing on $(0, \lambda_0)$ and increasing on (λ_0, ∞). The global minimum for $g_\alpha(\lambda)$ on $(0, \infty)$ is:

$$g_\alpha(\lambda_0) = A\left(\frac{\alpha B}{A}\right)^{\frac{1}{\alpha+1}} + B\left(\frac{A}{\alpha B}\right)^{\frac{\alpha}{\alpha+1}} = (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}}, \tag{2.5}$$

which proves (2.4). ☐

Lemma 2.4. *Let* $A, B > 0$ *and* $\alpha \in (0,1]$. *Consider the functions* $g_\alpha : (0,\infty) \to \mathbb{R}$ *and* $h_\alpha : (0,\infty) \to \mathbb{R}$ *defined by:*

$$\begin{cases} g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha \\ h_\alpha(\lambda) = \frac{2A}{\lambda} + \frac{B}{2^\alpha}\lambda^\alpha. \end{cases} \tag{2.6}$$

Define $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0,\infty)$. *Then for* $\lambda_1 \in (0,\infty)$ *we have:*

$$\begin{cases} \inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) < \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda) & \text{if} \quad 0 < \lambda_1 < 2\lambda_0 \\ \inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) = \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda) & \text{if} \quad \lambda_1 \geq 2\lambda_0. \end{cases} \tag{2.7}$$

Proof. In Lemma 2.3, we found that the global minimum for $g_\alpha(\lambda)$ is obtained for $\lambda = \lambda_0$. Similarly we find that the global minimum for $h_\alpha(\lambda)$ is obtained for $\lambda = 2\lambda_0$, and its value is equal to the minimal value of $g_\alpha(\lambda)$, i.e. $h_\alpha(2\lambda_0) = g_\alpha(\lambda_0)$.

The only solution of equation $g_\alpha(\lambda) = h_\alpha(\lambda)$, $\lambda \in (0,\infty)$, is:

$$\lambda_S = \left[\frac{A}{B(1-2^{-\alpha})}\right]^{\frac{1}{\alpha+1}},$$

and we can easily check that $\lambda_0 < \lambda_S < 2\lambda_0$. Thus, for $\lambda_1 < \lambda_0$ we have $g_\alpha(\lambda_1) < h_\alpha(\lambda_1)$ and $\inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) < \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda)$, and the rest of the proof is obvious. ☐

Proof of Theorem 2.1. Now we start proving our theorem using the identity:

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x [f'(s) - f'(a)]ds; \qquad a, x \in I \tag{2.8}$$

or, by changing x with a and a with x:

$$f(a) = f(x) + (a-x)f'(x) + \int_x^a [f'(s) - f'(x)]ds; \qquad a, x \in I. \tag{2.9}$$

Analogously, we have for $b \in I$:

$$f(b) = f(x) + (b-x)f'(x) + \int_x^b [f'(s) - f'(x)]ds; \qquad b, x \in I. \tag{2.10}$$

From (2.9) and (2.10) we obtain:

$$\begin{aligned} f(b) - f(a) = (b-a)f'(x) &+ \int_x^b [f'(s) - f'(x)]ds \\ &+ \int_a^x [f'(s) - f'(x)]ds; \quad a, b, x \in I \end{aligned} \tag{2.11}$$

and

$$f'(x) = \frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_x^b [f'(s) - f'(x)]ds - \frac{1}{b-a}\int_a^x [f'(s) - f'(x)]ds. \tag{2.12}$$

Assuming that $b > a$ we have the inequality:

$$|f'(x)| \leq \frac{|f(b) - f(a)|}{b - a} + \frac{1}{b - a}\left|\int_x^b |f'(s) - f'(x)|ds\right| + \frac{1}{b - a}\left|\int_a^x |f'(s) - f'(x)|ds\right|. \tag{2.13}$$

Since f' is of $\alpha - H$ Hölder type, then:

$$\begin{aligned}\left|\int_x^b |f'(s) - f'(x)|ds\right| &\leq H \cdot \left|\int_x^b |s - x|^\alpha ds\right| \\ &= H\int_x^b (s - x)^\alpha ds \\ &= \frac{H}{\alpha + 1}(b - x)^{\alpha+1}; \qquad b, x \in I,\ b > x\end{aligned} \tag{2.14}$$

$$\begin{aligned}\left|\int_a^x |f'(s) - f'(x)|ds\right| &\leq H \cdot \left|\int_a^x |s - x|^\alpha ds\right| \\ &= H\int_a^x (x - s)^\alpha ds \\ &= \frac{H}{\alpha + 1}(x - a)^{\alpha+1}; \qquad a, x \in I,\ a < x.\end{aligned} \tag{2.15}$$

From (2.13), (2.14) and (2.15) we deduce:

$$|f'(x)| \leq \frac{|f(b) - f(a)|}{b - a} + \frac{H}{(b - a)(\alpha + 1)}[(b - x)^{\alpha+1} + (x - a)^{\alpha+1}]; \quad a, b, x \in I,\ a < x < b. \tag{2.16}$$

Since $f \in L_\infty(I)$ then $|f(b) - f(a)| \leq 2 \cdot ||f||$. Using Lemma 2.2 we obviously get that:

$$|f'(x)| \leq \frac{2||f||}{b - a} + \frac{H}{\alpha + 1}(b - a)^\alpha; \qquad a, b, x \in I,\ a < x < b. \tag{2.17}$$

Denote $b - a = \lambda$. Since $a, b \in I$, $b > a$, we have $\lambda \in (0, m(I))$, and we can analyze the right-hand side of the inequality (2.17) as a function of variable λ. Thus we obtain:

$$|f'(x)| \leq \frac{2||f||}{\lambda} + \frac{H}{\alpha + 1}\lambda^\alpha = g_\alpha(\lambda) \tag{2.18}$$

for $x \in I$ and for every $\lambda \in (0, m(I))$.

Taking the infimum over $\lambda \in (0, m(I))$ in (2.18), we get:

$$|f'(x)| \leq \inf_{\lambda \in (0, m(I))} g_\alpha(\lambda). \tag{2.19}$$

If we take the supremum over $x \in I$ in (2.19) we conclude that

$$\sup_{x \in I} |f'(x)| = ||f'|| \leq \inf_{\lambda \in (0, m(I))} g_\alpha(\lambda). \tag{2.20}$$

Making use of Lemma 2.3 we obtain the desired result (2.1). □

Remark 2.5. Denote $\lambda_0 = \left[2\left(1+\frac{1}{\alpha}\right)\frac{||f||}{H}\right]^{\frac{1}{\alpha+1}}$. Comparing the results of Theorem A and Theorem 2.1 we can see that in the case of $m(I) \geq 2\lambda_0$ the estimated values for $||f'||$ in both theorems coincide. If $0 < m(I) < 2\lambda_0$ the estimated value for $||f'||$ given by (2.1) is better than the one given by (1.2). Namely, using Lemma 2.4 we have:

$$\frac{2||f||}{m(I)} + \frac{H}{\alpha+1}[m(I)]^{\alpha} < \frac{4||f||}{m(I)} + \frac{H}{2^{\alpha}(\alpha+1)}[m(I)]^{\alpha}; \quad m(I) \in (0, \lambda_0] \tag{2.21}$$

and

$$\left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} < \frac{4||f||}{m(I)} + \frac{H}{2^{\alpha}(\alpha+1)}[m(I)]^{\alpha}; \quad m(I) \in [\lambda_0, 2\lambda_0). \tag{2.22}$$

Remark 2.6. Let the conditions of Theorem 2.1 be fulfilled. Then a simple consequence of (2.11) is the following inequality:

$$|(b-a)f'(x) - f(b) + f(a)| \leq \frac{H}{\alpha+1}\left[(b-x)^{\alpha+1} + (x-a)^{\alpha+1}\right]; \quad a, b, x \in I, \ a < x < b.$$

This result is an extension of the result obtained by V.G. Avakumović and S. Aljančić in [2] (see also [3]).

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