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TRIPLE SOLUTIONS FOR A HIGHER-ORDER DIFFERENCE EQUATION

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Abstract

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Abstract

In this paper, we are concerned with the following nth difference equations

$$\Delta^n y(k-1) + f(k, y(k)) = 0, \quad k \in \{1, \dots, T\},$$

$$\Delta^i y(0) = 0, i = 0, 1, \dots, n-2, \quad \Delta^{n-2} y(T+1) = \alpha \Delta^{n-2} y(\xi),$$

where f is continuous, $n \geq 2$, $T \geq 3$ and $\xi \in \{2, \dots, T-1\}$ are three fixed positive integers, constant $\alpha > 0$ such that $\alpha\xi < T + 1$. Under some suitable conditions, we obtain the existence result of at least three positive solutions for the problem by using the Leggett-Williams fixed point theorem.

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Contents

1. Introduction

This paper deals with the following three-point discrete boundary value problem (BVP, for short):

$$\Delta^n y(k-1) + f(k, y(k)) = 0, \quad k \in \{1, \dots, T\}, \tag{1.1}$$

$$\Delta^i y(0) = 0,\ i = 0, 1, \dots, n-2, \quad \Delta^{n-2} y(T+1) = \alpha \Delta^{n-2} y(\xi), \tag{1.2}$$

where $\Delta y(k-1) = y(k) - y(k-1)$, $\Delta^n y(k-1) = \Delta^{n-1}(\Delta y(k-1))$, $k \in \{1, \dots, T\}$.

Throughout, we assume that the following conditions are satisfied:

(H_1) $T \geq 3$ and $\xi \in \{2, \dots, T-1\}$ are two fixed positive integers, $\alpha > 0$ such that $\alpha\xi < T+1$.

(H_2) $f \in C(\{1, \dots, T\} \times [0, +\infty), [0, +\infty))$ and $f(k, \cdot) \equiv 0$ does not hold on $\{1, \dots, \xi - 1\}$ and $\{\xi, \dots, T\}$.

In the few past years, there has been increasing interest in studying the existence of multiple positive solutions for differential and difference equations, for example, we refer the reader to [1] – [8].

Recently, Ma [9] studied the following second-order three-point boundary value problem

$$u'' + \lambda a(t) f(u) = 0,\ t \in (0,1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1), \tag{1.3}$$

by applying fixed-point index theorems and Leray-Schauder degree and upper and lower solutions. In the case $\lambda = 1$, under the conditions that f is superlinear or sublinear, Ma [10] considered the existence of at least one positive solution of problem (1.3) by using Krasnosel'skii's fixed-point theorem.

However, in [9] – [11], the author did not give the associate Green's function and exceptional work was carried out for higher order multi-point difference equations. In the current work, we give the associate Green's function and obtain the existence of multiple positive solutions for BVP (1.1) – (1.2) by employing the Leggett-Williams fixed point theorem. Our results are new and different from those in [9] – [11]. Particularly, we do not require the assumption that f is either superlinear or sublinear.

2. Background Definitions and Green's Function

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach space, which can be found in [3].

Let $\mathbf{N}$ be the nonnegative integers, we let $\mathbf{N}_{i,j} = \{k \in \mathbf{N} : i \leq k \leq j\}$ and $\mathbf{N}_p = \mathbf{N}_{0,p}$.

We say that y is a positive solution of BVP (1.1) – (1.2), if $y : \mathbf{N}_{T+n-1} \longrightarrow \mathbb{R}$, y satisfies (1.1) on $\mathbf{N}_{1,T}$, y fulfills (1.2) and y is nonnegative on $\mathbf{N}_{T+n-1}$ and positive on $\mathbf{N}_{n-1,T}$.

Definition 2.1. *Let E be a Banach space, a nonempty closed set $K \subset E$ is said to be a cone provided that*

(i) if $x \in K$ and $\lambda \geq 0$ then $\lambda x \in K$;

(ii) if $x \in K$ and $-x \in K$ then $x = 0$.

If $K \subset E$ is a cone, we denote the order induced by K on E by $\leq$. For $x, y \in K$, we write $x \leq y$ if and only if $y - x \in K$.

Definition 2.2. *A map h is a nonnegative continuous concave functional on the cone K which is convex, provided that*

(i) $h : K \longrightarrow [0, \infty)$ is continuous;

(ii) $h(tx + (1-t)y) \geq th(x) + (1-t)h(y)$ for all $x, y \in K$ and $0 \leq t \leq 1$.

Now we shall denote

$$K_c = \{y \in K : \|y\| < c\}$$

and

$$K(h, a, b) = \{y \in K : h(y) \geq a, \|y\| \leq b\},$$

where $\|\cdot\|$ is the maximum norm.

Next we shall state the fixed point theorem due to Leggett-Williams [12] also see [3].

Theorem 2.1. *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume that h is a nonnegative continuous concave functional on K such that $h(y) \leq \|y\|$ for all $y \in \overline{K_c}$, and let $S : \overline{K_c} \longrightarrow \overline{K_c}$ be a completely continuous operator. Suppose that there exist $0 < a < b < d \leq c$ such that*

(A_1) $\{y \in K(h, b, d) : h(y) > b\} \neq \emptyset$ *and* $h(Sy) > b$ *for all* $y \in K(h, b, d)$*;*

(A_2) $\|Sy\| < a$ *for* $\|y\| < a$*;*

(A_3) $h(Sy) > b$ *for all* $y \in K(h, b, c)$ *with* $\|Sy\| > d$*.*

Then S has at least three fixed points y_1, y_2 and y_3 in $\overline{K_c}$ such that $\|y_1\| < a$, $h(y_2) > b$ and $\|y_3\| > a$ with $h(y_3) < b$.

In the following, we assume that the function $G(k, l)$ is the Green's function of the problem $-\Delta^n y(k-1) = 0$ with the boundary condition (1.2).

It is clear that (see [3])

$$g(k, l) = \Delta^{n-2} G(k, l), \text{ (with respect to } k)$$

is the Green's function of the problem $-\Delta^2 y(k-1) = 0$ with the boundary condition

$$y(0) = 0, \qquad y(T+1) = \alpha y(\xi). \tag{2.1}$$

We shall give the Green's function of the problem $-\Delta^2 y(k-1) = 0$ with the boundary condition (2.1).

Lemma 2.2. *The problem*

$$\Delta^2 y(k-1) + u(k) = 0, \qquad k \in \mathbf{N}_{1,T}, \tag{2.2}$$

with the boundary condition (2.1) has the unique solution

$$y(k) = -\sum_{l=1}^{k-1}(k-l)u(l) + \frac{k}{T+1-\alpha\xi}\sum_{l=1}^{T}(T+1-l)u(l) - \frac{\alpha k}{T+1-\alpha\xi}\sum_{l=1}^{\xi-1}(\xi-l)u(l), \quad k \in \mathbf{N}_{T+1}. \tag{2.3}$$

Proof. From (2.2), one has

$$\begin{aligned}
\Delta y(k) - \Delta y(k-1) &= -u(k),\\
\Delta y(k-1) - \Delta y(k-2) &= -u(k-1),\\
&\vdots\\
\Delta y(1) - \Delta y(0) &= -u(1).
\end{aligned}$$

We sum the above equalities to obtain

$$\Delta y(k) = \Delta y(0) - \sum_{l=1}^{k} u(l), \quad k \in \mathbf{N}_T,$$

here and in the following, we denote $\sum_{l=p}^{q} u(l) = 0$, if $p > q$. Similarly, we sum the equalities from 0 to k and change the order of summation to obtain

$$\begin{aligned} y(k+1) &= y(0) + (k+1)\Delta y(0) - \sum_{l=1}^{k} \sum_{j=1}^{l} u(j) \\ &= y(0) + (k+1)\Delta y(0) - \sum_{l=1}^{k} (k+1-l)u(l), \quad k \in \mathbf{N}_T, \end{aligned}$$

i.e.,

$$y(k) = y(0) + k\Delta y(0) - \sum_{l=1}^{k-1} (k-l)u(l), \quad k \in \mathbf{N}_{T+1}. \tag{2.4}$$

By using the boundary condition (2.1), we have

$$\Delta y(0) = \frac{1}{T+1-\alpha\xi} \sum_{l=1}^{T} (T+1-l)u(l) - \frac{\alpha}{T+1-\alpha\xi} \sum_{l=1}^{\xi-1} (\xi - l)u(l). \tag{2.5}$$

By (2.4) and (2.5), we have shown that (2.3) holds. □

Lemma 2.3. *The function*

$$g(k,l)=\begin{cases}\frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi}, & l\in \mathbf{N}_{1,k-1}\bigcap \mathbf{N}_{1,\xi-1};\\ \frac{l(T+1-k)+\alpha\xi(k-l)}{T+1-a\xi}, & l\in \mathbf{N}_{\xi,k-1};\\ \frac{k[T+1-l-\alpha(\xi-l)]}{T+1-\alpha\xi}, & l\in \mathbf{N}_{k,\xi-1};\\ \frac{k(T+1-l)}{T+1-\alpha\xi}, & l\in \mathbf{N}_{k,T}\bigcap \mathbf{N}_{\xi,T}.\end{cases} \tag{2.6}$$

is the Green's function of the following problem

$$-\Delta^2 y(k-1)=0,\ \ k\in \mathbf{N}_{1,T}, \tag{2.7}$$

$$y(0)=0,\ \ y(T+1)=\alpha y(\xi). \tag{2.1}$$

Proof. We shall divide the proof into the following two steps.

Step 1. We suppose $k<\xi$. Then the unique solution of problem (2.7), (2.1) can be written as

$$\begin{aligned}y(k)=&-\sum_{l=1}^{k-1}(k-l)u(l)+\frac{k}{T+1-\alpha\xi}\sum_{l=1}^{k-1}(T+1-l)u(l)\\ &+\frac{k}{T+1-\alpha\xi}\left[\sum_{l=k}^{\xi-1}(T+1-l)u(l)+\sum_{l=\xi}^{T}(T+1-l)u(l)\right]\\ &-\frac{\alpha k}{T+1-\alpha\xi}\left[\sum_{l=1}^{k-1}(\xi-l)u(l)+\sum_{l=k}^{\xi-1}(\xi-l)u(l)\right]\end{aligned}$$

$$= \sum_{l=1}^{k-1} \frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi} u(l) + \sum_{l=k}^{\xi-1} \frac{k[T+1-l-\alpha(\xi-l)]}{T+1-\alpha\xi} u(l) + \sum_{l=\xi}^{T} \frac{k(T+1-l)}{T+1-\alpha\xi} u(l)$$

$$= \sum_{l=1}^{T} g(k,l)u(l).$$

Step 2. We suppose $k \geq \xi$. Then the unique solution of problem (2.7), (2.1) can be written as

$$y(k) = -\left[\sum_{l=1}^{\xi-1}(k-l)u(l) + \sum_{l=\xi}^{k-1}(k-l)u(l)\right] + \frac{k}{T+1-\alpha\xi}\left[\sum_{l=1}^{\xi-1}(T+1-l)u(l) + \sum_{l=\xi}^{k-1}(T+1-l)u(l) + \sum_{l=k}^{T}(T+1-l)u(l)\right] - \frac{\alpha k}{T+1-\alpha\xi}\sum_{l=1}^{\xi-1}(\xi-l)u(l)$$

$$= \sum_{l=1}^{\xi-1} \frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi} u(l) + \sum_{l=\xi}^{k-1} \frac{l(T+1-k)+\alpha\xi(k-l)}{T+1-\alpha\xi} u(l) + \sum_{l=k}^{T} \frac{k(T+1-l)}{T+1-\alpha\xi} u(l)$$

$$= \sum_{l=1}^{T} g(k,l)u(l).$$

Thus the unique solution of problem (2.7), (2.1) can be written as $y(k) = \sum_{l=1}^{T} g(k,l)u(l)$. □

We observe that the condition $\alpha\xi < T+1$ implies that $g(k,l)$ is nonnegative on $\mathbf{N}_{T+1} \times \mathbf{N}_{1,T}$, and positive on $\mathbf{N}_{1,T} \times \mathbf{N}_{1,T}$. From (2.3), we have

$$y(k) = \sum_{l=1}^{T} g(k,l)u(l),$$

where

$$g(k,l) := (T+1-\alpha\xi)^{-1}\Big(k(T+1-l) - (k-l)(T+1-\alpha\xi)\chi_{[1,k-1]}(l) - \alpha k(\xi - l)\chi_{[1,\xi-1]}(l)\Big).$$

This is a positive function, which means that the finite set

$$\{g(k,l)/g(k,k) : k,l = 1,2,\dots,T\}$$

takes positive values. Let M_1, M_2 be its minimum and maximum values, respectively.

3. Existence of Triple Solutions

In the following, we denote

$$m = \min_{k \in \mathbf{N}_{\xi,T}} \sum_{l=\xi}^{T} g(k,l), \qquad M = \max_{k \in \mathbf{N}_{T+1}} \sum_{l=1}^{T} g(k,l)$$

and

$$\widetilde{m} = \min_{k \in \mathbf{N}_{\xi,T}} g(k,k), \qquad \widetilde{M} = \max_{k \in \mathbf{N}_{T+1}} g(k,k).$$

Then $0 < m < M$, $0 < \widetilde{m} < \widetilde{M}$.

Let E be the Banach space defined by

$$E = \{y : \mathbf{N}_{T+n-1} \longrightarrow \mathbb{R},\ \Delta^i y(0) = 0,\ i = 0, 1, \dots, n-2\}.$$

Define

$$K = \left\{ y \in E : \Delta^{n-2} y(k) \geq 0 \text{ for } \ k \in \mathbf{N}_{T+1} \text{ and } \min_{k \in \mathbf{N}_{\xi,T}} \Delta^{n-2} y(k) \geq \sigma \|y\| \right\}$$

where $\sigma = \frac{M_1 \widetilde{m}}{M_2 \widetilde{M}} \in (0,1)$, $\|y\| = \max_{k \in \mathbf{N}_{T+1}} |\Delta^{n-2} y(k)|$. It is clear that K is a cone in E.

Finally, let the nonnegative continuous concave functional $h : K \longrightarrow [0, \infty)$ be defined by

$$h(y) = \min_{k \in \mathbf{N}_{\xi,T}} \Delta^{n-2} y(k), \qquad y \in K.$$

Note that for $y \in K$, $h(y) \leq \|y\|$.

Remark 1. *If $y \in K$, $\|y\| \le c$, then*

$$0 \le y(k) \le qc, \qquad k \in \mathbf{N}_{T+n-1},$$

where

$$q = q(n,T) = \frac{(T+n-1)(T+n)\cdots(T+2n-4)}{(n-2)!}.$$

In fact, if $y \in K$, $\|y\| \le c$, then $0 \le \Delta^{n-2}y(k) \le c$, $k \in \mathbf{N}_{T+1}$, i.e.,

$$0 \le \Delta(\Delta^{n-3}y(k)) = \Delta^{n-3}y(k+1) - \Delta^{n-3}y(k) \le c.$$

Then one has

$$\begin{aligned} 0 \le \Delta^{n-3}y(1) - \Delta^{n-3}y(0) \le c, \\ 0 \le \Delta^{n-3}y(2) - \Delta^{n-3}y(1) \le c, \\ \vdots \\ 0 \le \Delta^{n-3}y(k) - \Delta^{n-3}y(k-1) \le c. \end{aligned}$$

We sum the above inequalities to obtain

$$0 \le \Delta^{n-3}y(k) \le kc, \qquad k \in \mathbf{N}_{T+2}.$$

Similarly, we have

$$0 \le \Delta^{n-4}y(k) \le \left(\sum_{i=1}^{k} i\right) c = \frac{k(k+1)}{2!}c, \qquad k \in \mathbf{N}_{T+3}.$$

By using the induction method, one has

$$0 \le y(k) \le \frac{k(k+1)\cdots(k+n-3)}{(n-2)!}c, \qquad k \in \mathbf{N}_{T+n-1}.$$

Then

$$0 \le y(k) \le \frac{(T+n-1)(T+n)\cdots(T+2n-4)}{(n-2)!}c = qc, \qquad k \in \mathbf{N}_{T+n-1}.$$

Theorem 3.1. *Assume that there exist constants* a, b, c *such that* $0 < a < b < c \cdot \min\left\{\sigma, \frac{m}{M}\right\}$ *and satisfy*

(H_3) $f(k,y) \le \frac{c}{M}$, $(k,y) \in [0, T+n-1] \times [0, qc]$,

(H_4) $f(k,y) < \frac{a}{M}$, $(k,y) \in [0, T+n-1] \times [0, qa]$,

(H_5) *There exists some* $l_0 \in [n-2, T+n-1]$*, such that* $f(k,y) \ge \frac{b}{m_0}$, $(k,y) \in [n-2, T+n-1] \times \left[b, \frac{qb}{\sigma}\right]$*, where* $m_0 = \min_{k,l \in \mathbf{N}_T} g(k,l) > 0$.

Then BVP (1.1) – (1.2) has at least three positive solutions y_1, y_2 *and* y_3*, such that*

$$\|y_1\| < a, \quad h(y_2) > b, \tag{3.1}$$

and

$$\|y_3\| > a \quad \text{with} \quad h(y_3) < b. \tag{3.2}$$

Proof. Let the operator $S : K \longrightarrow E$ be defined by

$$(Sy)(k) = \sum_{l=1}^{T} G(k,l) f(l, y(l)), \qquad k \in \mathbf{N}_{T+n-1}.$$

It follows that

$$\Delta^{n-2}(Sy)(k) = \sum_{l=1}^{T} g(k,l)f(l,y(l)), \text{ for } \quad k \in \mathbf{N}_{T+1}. \tag{3.3}$$

We shall now show that the operator S maps K into itself. For this, let $y \in K$, from (H_1), (H_2), one has

$$\Delta^{n-2}(Sy)(k) = \sum_{l=1}^{T} g(k,l)f(l,y(l)) \geq 0, \text{ for } \quad k \in \mathbf{N}_{T+1}, \tag{3.4}$$

and

$$\begin{aligned} \Delta^{n-2}(Sy)(k) &= \sum_{l=1}^{T} g(k,l)f(l,y(l)) \\ &\leq M_2 \sum_{l=1}^{T} g(k,k)f(l,y(l)) \\ &\leq M_2\widetilde{M} \sum_{l=1}^{T} f(l,y(l)), \qquad \text{for} \qquad k \in \mathbf{N}_{T+1}. \end{aligned}$$

Thus

$$\|Sy\| \leq M_2\widetilde{M} \sum_{l=1}^{T} f(l,y(l)).$$

From (H_1), (H_2), and (3.3), for $k \in \mathbf{N}_{\xi,T}$, we have

$$\begin{aligned} \Delta^{n-2}(Sy)(k) &\geq M_1 \sum_{l=1}^{T} g(k,k)f(l,y(l)) \\ &\geq M_1\widetilde{m} \sum_{l=1}^{T} f(l,y(l)) \geq \frac{M_1\widetilde{m}}{M_2\widetilde{M}}\|Sy\| = \sigma\|Sy\|. \end{aligned}$$

Subsequently

$$\min_{k\in\mathbf{N}_{\xi,T}} \Delta^{n-2}(Sy)(k) \geq \sigma\|Sy\|. \tag{3.5}$$

From (3.4) and (3.5), we obtain $Sy \in K$. Hence $S(K) \subseteq K$. Also standard arguments yield that $S : K \longrightarrow K$ is completely continuous.

We now show that all of the conditions of Theorem 2.1 are fulfilled. For all $y \in \overline{K_c}$, we have $\|y\| \leq c$. From assumption (H_3), we get

$$\begin{aligned} \|Sy\| &= \max_{k\in\mathbf{N}_{T+1}} |\Delta^{n-2}(Sy)(k)| \\ &= \max_{k\in\mathbf{N}_{T+1}} \left|\sum_{l=1}^{T} g(k,l) f(l,y(l))\right| \\ &\leq \frac{c}{M} \max_{k\in\mathbf{N}_{T+1}} \sum_{l=1}^{T} g(k,l) = c. \end{aligned}$$

Hence $S : \overline{K_c} \longrightarrow \overline{K_c}$.

Similarly, if $y \in K_a$, then assumption (H_4) yields $f(k,y) < \frac{a}{M}$, for $k \in \mathbf{N}_{T+1}$. As in the argument above, we can show $S : \overline{K_a} \longrightarrow K_a$. Therefore, condition (A_2) of Theorem 2.1 is satisfied.

Now we prove that condition (A_1) of Theorem 2.1 holds. Let

$$y^*(k) = \frac{k(k+1)\cdots(k+n-3)b}{(n-2)!\sigma}, \quad \text{for} \qquad k \in \mathbf{N}_{\xi,T}.$$

Then we can show that $y^* \in K\left(h, b, \frac{qb}{\sigma}\right)$ and $h(y^*) \geq \frac{b}{\sigma} > b$. So

$$\left\{y \in K\left(h, b, \frac{b}{\sigma}\right) : h(y) > b\right\} \neq \emptyset.$$

From assumptions (H_2) and (H_5), one has

$$\begin{aligned} h(Sy) &= \min_{k \in \mathbf{N}_{\xi,T}} \sum_{l=1}^{T} g(k,l) f(l, y(l)) \\ &> \min_{k \in \mathbf{N}_{\xi,T}} \sum_{l=\xi}^{T} g(k,l) f(l, y(l)) \\ &\geq \min_{k \in \mathbf{N}_{\xi,T}} g(k,l_0) f(l_0, y(l_0)) \\ &\geq \frac{b}{m_0} \min_{k \in \mathbf{N}_{\xi,T}} g(k,l) \geq b. \end{aligned}$$

This shows that condition (A_1) of Theorem 2.1 is satisfied.

Finally, suppose that $y \in K(h, b, c)$ with $\|Sy\| > \frac{b}{\sigma}$, then

$$h(Sy) = \min_{k \in \mathbf{N}_{\xi,T}} \Delta^{n-2}(Sy)(k) \geq \sigma \|Sy\| > b.$$

Thus, condition (A_3) of Theorem 2.1 is also satisfied. Therefore, Theorem 2.1 implies that BVP (1.1) – (1.2) has at least three positive solutions y_1, y_2, y_3 described by (3.1) and (3.2). □

Corollary 3.2. *Suppose that there exist constants*

$$0 < a_1 < b_1 < c_1 \cdot \min\left\{\sigma, \frac{m}{M}\right\} < a_2 < b_2 < c_2 \cdot \min\left\{\sigma, \frac{m}{M}\right\} < \cdots < a_p,$$

p *is a positive integer, such that the following conditions are satisfied:*

(H_7) $f(k,y) < \frac{a_i}{M}$, $(k,y) \in [0, T+n-1] \times [0, qa_i]$, $i \in \mathbf{N}_{1,p}$;

(H_8) *There exist* $l_{i0} \in [n-2, T+n-1]$, *such that* $f(k,y) \geq \frac{qb_i}{m_0}$, $(k,y) \in [n-2, T+n-1] \times [b_i, \frac{qb_i}{\sigma}]$, $i \in \mathbf{N}_{1,p-1}$.

Then BVP (1.1) – (1.2) has at least $2p-1$ *positive solutions.*

Proof. When $p = 1$, from condition (H_7), we show $S : \overline{K_{a_1}} \longrightarrow K_{a_1} \subseteq \overline{K_{a_1}}$. By using the Schauder fixed point theorem, we show that BVP (1.1) – (1.2) has at least one fixed point $y_1 \in \overline{K_{a_1}}$. When $p = 2$, it is clear that Theorem 3.1 holds (with $c_1 = a_2$). Then we can obtain BVP (1.1) – (1.2) has at least three positive solutions y_1, y_2 and y_3, such that $\|y_1\| < a_1$, $h(y_2) > b_1$, $\|y_3\| > a_1$, with $h(y_3) < b_1$. Following this way, we finish the proof by the induction method. The proof is completed. □

If the case $n = 2$, similar to the proof of Theorem 3.1, we obtain the following result.

Corollary 3.3. *Assume that there exist constants* a, b, c *such that* $0 < a < b < c \cdot \min\left\{\sigma, \frac{m}{M}\right\}$ *and satisfy*

(H_9) $f(k,y) \leq \frac{c}{M}$, $(k,y) \in [0, T+n-1] \times [0, c]$,

(H_{10}) $f(k,y) < \frac{a}{M}$, $(k,y) \in [0, T+n-1] \times [0,a]$,

(H_{11}) $f(k,y) \geq \frac{b}{m}$, $(k,y) \in [\xi, T+n-1] \times [b, \frac{b}{\sigma}]$.

Then BVP (1.1) – (1.2) has at least three positive solutions y_1, y_2 and y_3, satisfying (3.1) and (3.2).

Finally, we give an example to illustrate our main result.

Example 3.1. *Consider the following second order third point boundary value problem*

$$\Delta^2 y(k-1) + f(k,y) = 0, \quad k \in \mathbf{N}_{1,6}, \tag{3.6}$$

$$y(0) = 0, \quad y(7) = \frac{7}{9} y(3), \tag{3.7}$$

where $f(k,y) = \frac{100}{k+100} a(y)$, and

$$a(y) = \begin{cases} \frac{1}{720} + \sin^8 y, & \textit{if } y \in \left[0, \frac{1}{30}\right]; \\ \frac{1}{720} + 6\left(y - \frac{1}{30}\right) + \sin^8 y, & \textit{if } y \in \left[\frac{1}{30}, 3\right]; \\ \frac{1}{720} + \frac{89}{5} + \frac{\sin^2(y-3)}{2} + \sin^8 y, & \textit{if } y \in [3, 360]. \end{cases}$$

Then $T = 6$, $n = 3$, $\alpha = \frac{7}{9} < 1$, $T + 1 - \alpha n = \frac{14}{3} > 0$. Then the conditions

(H_1), (H_2) *are satisfied, and the function*

$$G(k,l) = \frac{3}{14}\begin{cases} \frac{l(42-2k)}{9}, & l \in \mathbf{N}_{1,k-1} \bigcap \mathbf{N}_{1,2}; \\ \frac{3l(7-k)+7(k-l)}{3}, & l \in \mathbf{N}_{3,k-1}; \\ \frac{k(42-2l)}{9}, & l \in \mathbf{N}_{k,2}; \\ k(7-l), & l \in \mathbf{N}_{k,6} \bigcap \mathbf{N}_{3,6}, \end{cases}$$

is the Green's function of the problem $-\Delta^2 y(k-1) = 0$, $k \in \mathbf{N}_{1,6}$ *with* (3.7).

Thus we can compute $m = \frac{27}{2}$, $M = 18$, $\widetilde{m} = \frac{9}{7}$, $\widetilde{M} = \frac{18}{7}$, $M_1 = \frac{2}{9}$, $M_2 = 9$, $\sigma = \frac{1}{81} < \frac{m}{M} = \frac{3}{4}$. *We choose that* $a = \frac{1}{35}$, $b = \frac{1}{10}$, $c = 360$, *consequently,*

$$\begin{aligned} f(k,y) &= \frac{100}{k+100}a(y) \\ &\le a(y) \le \begin{cases} \frac{1}{720} + 1 < 20 = \frac{c}{M}, & (k,y) \in [0,7] \times \left[0, \frac{1}{30}\right]; \\ \frac{1}{720} + 6\left(3 - \frac{1}{30}\right) + 1 < 20 = \frac{c}{M}, & (k,y) \in [0,7] \times \left[\frac{1}{30}, 3\right]; \\ \frac{1}{720} + \frac{89}{5} + \frac{3}{2} < 20 = \frac{c}{M}, & (k,y) \in [0,7] \times [3, 360]. \end{cases} \end{aligned}$$

Thus

$$f(k,y) \le \frac{c}{M}, \qquad (k,y) \in [0,7] \times [0,360];$$

and

$$f(k,y) \le \frac{1}{720} + \sin^8 y < \frac{1}{630} = \frac{a}{M}, \qquad (k,y) \in [0,7] \times \left[0, \frac{1}{35}\right];$$

Triple Solutions for a Higher-order Difference Equation

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$$f(k,y) \geq \frac{100}{107}\left[\frac{1}{720} + 6\left(\frac{1}{10} - \frac{1}{30}\right) + \sin^8 y\right] \geq \frac{1}{135} = \frac{b}{m},$$
$$(k,y) \in [3,7] \times \left[\frac{1}{27}, 3\right].$$

That is to say, all the conditions of Corollary 3.3 are satisfied. Then the boundary value problem (3.6), (3.7) has at least three positive solutions y_1, y_2 *and* y_3*, such that*

$$y_1(k) < \frac{1}{35}, \quad \textit{for } k \in \mathbf{N}_7, \quad y_2(k) > \frac{1}{27}, \quad \textit{for } k \in \mathbf{N}_{3,7},$$

and

$$\max_{k \in \mathbf{N}_{1,7}} y_3(k) > \frac{1}{35}, \quad \textit{for } k \in \mathbf{N}_7 \ \textit{ with } \min_{k \in \mathbf{N}_{3,7}} y_3(k) < \frac{1}{27}.$$

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