

## TRIPLE SOLUTIONS FOR A HIGHER-ORDER DIFFERENCE EQUATION

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## Abstract

In this paper, we are concerned with the following  $n$ th difference equations

$$\Delta^n y(k-1) + f(k, y(k)) = 0, \quad k \in \{1, \dots, T\},$$

$$\Delta^i y(0) = 0, i = 0, 1, \dots, n-2, \quad \Delta^{n-2} y(T+1) = \alpha \Delta^{n-2} y(\xi),$$

where  $f$  is continuous,  $n \geq 2$ ,  $T \geq 3$  and  $\xi \in \{2, \dots, T-1\}$  are three fixed positive integers, constant  $\alpha > 0$  such that  $\alpha\xi < T+1$ . Under some suitable conditions, we obtain the existence result of at least three positive solutions for the problem by using the Leggett-Williams fixed point theorem.

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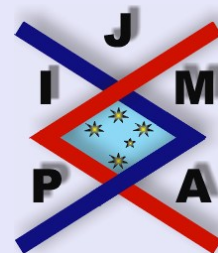
*Key words:* Discrete three-point boundary value problem; Multiple solutions; Green's function; Cone; Fixed point.

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# 1. Introduction

This paper deals with the following three-point discrete boundary value problem (BVP, for short):

$$(1.1) \quad \Delta^n y(k-1) + f(k, y(k)) = 0, \quad k \in \{1, \dots, T\},$$

$$(1.2) \quad \Delta^i y(0) = 0, \quad i = 0, 1, \dots, n-2, \quad \Delta^{n-2} y(T+1) = \alpha \Delta^{n-2} y(\xi),$$

where  $\Delta y(k-1) = y(k) - y(k-1)$ ,  $\Delta^n y(k-1) = \Delta^{n-1}(\Delta y(k-1))$ ,  $k \in \{1, \dots, T\}$ .

Throughout, we assume that the following conditions are satisfied:

( $H_1$ )  $T \geq 3$  and  $\xi \in \{2, \dots, T-1\}$  are two fixed positive integers,  $\alpha > 0$  such that  $\alpha\xi < T+1$ .

( $H_2$ )  $f \in C(\{1, \dots, T\} \times [0, +\infty), [0, +\infty))$  and  $f(k, \cdot) \equiv 0$  does not hold on  $\{1, \dots, \xi-1\}$  and  $\{\xi, \dots, T\}$ .

In the few past years, there has been increasing interest in studying the existence of multiple positive solutions for differential and difference equations, for example, we refer the reader to [1] – [8].

Recently, Ma [9] studied the following second-order three-point boundary value problem

$$(1.3) \quad u'' + \lambda a(t)f(u) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1),$$



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by applying fixed-point index theorems and Leray-Schauder degree and upper and lower solutions. In the case  $\lambda = 1$ , under the conditions that  $f$  is superlinear or sublinear, Ma [10] considered the existence of at least one positive solution of problem (1.3) by using Krasnosel'skii's fixed-point theorem.

However, in [9] – [11], the author did not give the associate Green's function and exceptional work was carried out for higher order multi-point difference equations. In the current work, we give the associate Green's function and obtain the existence of multiple positive solutions for BVP (1.1) – (1.2) by employing the Leggett-Williams fixed point theorem. Our results are new and different from those in [9] – [11]. Particularly, we do not require the assumption that  $f$  is either superlinear or sublinear.



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## 2. Background Definitions and Green's Function

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach space, which can be found in [3].

Let  $\mathbf{N}$  be the nonnegative integers, we let  $\mathbf{N}_{i,j} = \{k \in \mathbf{N} : i \leq k \leq j\}$  and  $\mathbf{N}_p = \mathbf{N}_{0,p}$ .

We say that  $y$  is a positive solution of BVP (1.1) – (1.2), if  $y : \mathbf{N}_{T+n-1} \rightarrow \mathbb{R}$ ,  $y$  satisfies (1.1) on  $\mathbf{N}_{1,T}$ ,  $y$  fulfills (1.2) and  $y$  is nonnegative on  $\mathbf{N}_{T+n-1}$  and positive on  $\mathbf{N}_{n-1,T}$ .

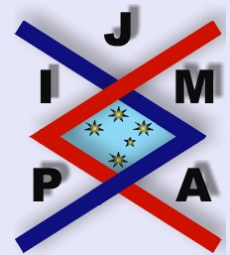
**Definition 2.1.** Let  $E$  be a Banach space, a nonempty closed set  $K \subset E$  is said to be a cone provided that

- (i) if  $x \in K$  and  $\lambda \geq 0$  then  $\lambda x \in K$ ;
- (ii) if  $x \in K$  and  $-x \in K$  then  $x = 0$ .

If  $K \subset E$  is a cone, we denote the order induced by  $K$  on  $E$  by  $\leq$ . For  $x, y \in K$ , we write  $x \leq y$  if and only if  $y - x \in K$ .

**Definition 2.2.** A map  $h$  is a nonnegative continuous concave functional on the cone  $K$  which is convex, provided that

- (i)  $h : K \rightarrow [0, \infty)$  is continuous;
- (ii)  $h(tx + (1 - t)y) \geq th(x) + (1 - t)h(y)$  for all  $x, y \in K$  and  $0 \leq t \leq 1$ .



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Now we shall denote

$$K_c = \{y \in K : \|y\| < c\}$$

and

$$K(h, a, b) = \{y \in K : h(y) \geq a, \|y\| \leq b\},$$

where  $\|\cdot\|$  is the maximum norm.

Next we shall state the fixed point theorem due to Leggett-Williams [12] also see [3].

**Theorem 2.1.** *Let  $E$  be a Banach space, and let  $K \subset E$  be a cone in  $E$ . Assume that  $h$  is a nonnegative continuous concave functional on  $K$  such that  $h(y) \leq \|y\|$  for all  $y \in \overline{K_c}$ , and let  $S : \overline{K_c} \rightarrow \overline{K_c}$  be a completely continuous operator. Suppose that there exist  $0 < a < b < d \leq c$  such that*

(A<sub>1</sub>)  $\{y \in K(h, b, d) : h(y) > b\} \neq \emptyset$  and  $h(Sy) > b$  for all  $y \in K(h, b, d)$ ;

(A<sub>2</sub>)  $\|Sy\| < a$  for  $\|y\| < a$ ;

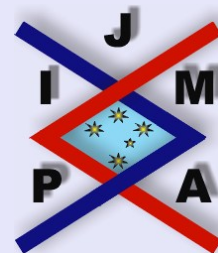
(A<sub>3</sub>)  $h(Sy) > b$  for all  $y \in K(h, b, c)$  with  $\|Sy\| > d$ .

Then  $S$  has at least three fixed points  $y_1, y_2$  and  $y_3$  in  $\overline{K_c}$  such that  $\|y_1\| < a$ ,  $h(y_2) > b$  and  $\|y_3\| > a$  with  $h(y_3) < b$ .

In the following, we assume that the function  $G(k, l)$  is the Green's function of the problem  $-\Delta^n y(k-1) = 0$  with the boundary condition (1.2).

It is clear that (see [3])

$$g(k, l) = \Delta^{n-2}G(k, l), \text{ (with respect to } k\text{)}$$




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is the Green's function of the problem  $-\Delta^2 y(k-1) = 0$  with the boundary condition

$$(2.1) \quad y(0) = 0, \quad y(T+1) = \alpha y(\xi).$$

We shall give the Green's function of the problem  $-\Delta^2 y(k-1) = 0$  with the boundary condition (2.1).

**Lemma 2.2.** *The problem*

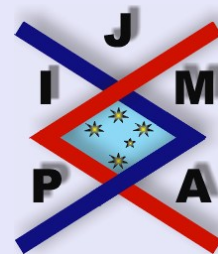
$$(2.2) \quad \Delta^2 y(k-1) + u(k) = 0, \quad k \in \mathbf{N}_{1,T},$$

*with the boundary condition (2.1) has the unique solution*

$$(2.3) \quad y(k) = -\sum_{l=1}^{k-1} (k-l)u(l) + \frac{k}{T+1-\alpha\xi} \sum_{l=1}^T (T+1-l)u(l) - \frac{\alpha k}{T+1-\alpha\xi} \sum_{l=1}^{\xi-1} (\xi-l)u(l), \quad k \in \mathbf{N}_{T+1}.$$

*Proof.* From (2.2), one has

$$\begin{aligned} \Delta y(k) - \Delta y(k-1) &= -u(k), \\ \Delta y(k-1) - \Delta y(k-2) &= -u(k-1), \\ &\vdots \\ \Delta y(1) - \Delta y(0) &= -u(1). \end{aligned}$$



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We sum the above equalities to obtain

$$\Delta y(k) = \Delta y(0) - \sum_{l=1}^k u(l), \quad k \in \mathbf{N}_T,$$

here and in the following, we denote  $\sum_{l=p}^q u(l) = 0$ , if  $p > q$ . Similarly, we sum the equalities from 0 to  $k$  and change the order of summation to obtain

$$\begin{aligned} y(k+1) &= y(0) + (k+1)\Delta y(0) - \sum_{l=1}^k \sum_{j=1}^l u(j) \\ &= y(0) + (k+1)\Delta y(0) - \sum_{l=1}^k (k+1-l)u(l), \quad k \in \mathbf{N}_T, \end{aligned}$$

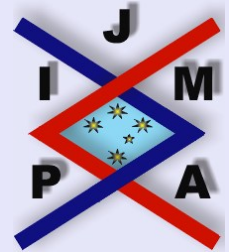
i.e.,

$$(2.4) \quad y(k) = y(0) + k\Delta y(0) - \sum_{l=1}^{k-1} (k-l)u(l), \quad k \in \mathbf{N}_{T+1}.$$

By using the boundary condition (2.1), we have

$$(2.5) \quad \Delta y(0) = \frac{1}{T+1-\alpha\xi} \sum_{l=1}^T (T+1-l)u(l) - \frac{\alpha}{T+1-\alpha\xi} \sum_{l=1}^{\xi-1} (\xi-l)u(l).$$

By (2.4) and (2.5), we have shown that (2.3) holds.  $\square$



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**Lemma 2.3.** *The function*

$$(2.6) \quad g(k, l) = \begin{cases} \frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi}, & l \in \mathbf{N}_{1,k-1} \cap \mathbf{N}_{1,\xi-1}; \\ \frac{l(T+1-k)+\alpha\xi(k-l)}{T+1-\alpha\xi}, & l \in \mathbf{N}_{\xi,k-1}; \\ \frac{k[T+1-l-\alpha(\xi-l)]}{T+1-\alpha\xi}, & l \in \mathbf{N}_{k,\xi-1}; \\ \frac{k(T+1-l)}{T+1-\alpha\xi}, & l \in \mathbf{N}_{k,T} \cap \mathbf{N}_{\xi,T}. \end{cases}$$

is the Green's function of the following problem

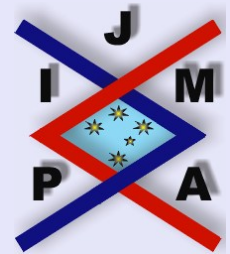
$$(2.7) \quad -\Delta^2 y(k-1) = 0, \quad k \in \mathbf{N}_{1,T},$$

$$(2.1) \quad y(0) = 0, \quad y(T+1) = \alpha y(\xi).$$

*Proof.* We shall divide the proof into the following two steps.

**Step 1.** We suppose  $k < \xi$ . Then the unique solution of problem (2.7), (2.1) can be written as

$$\begin{aligned} y(k) = & -\sum_{l=1}^{k-1} (k-l)u(l) + \frac{k}{T+1-\alpha\xi} \sum_{l=1}^{k-1} (T+1-l)u(l) \\ & + \frac{k}{T+1-\alpha\xi} \left[ \sum_{l=k}^{\xi-1} (T+1-l)u(l) + \sum_{l=\xi}^T (T+1-l)u(l) \right] \\ & - \frac{\alpha k}{T+1-\alpha\xi} \left[ \sum_{l=1}^{k-1} (\xi-l)u(l) + \sum_{l=k}^{\xi-1} (\xi-l)u(l) \right] \end{aligned}$$



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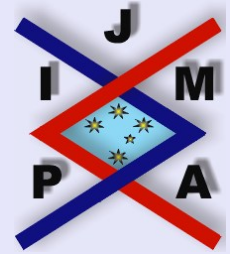
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$$\begin{aligned}
&= \sum_{l=1}^{k-1} \frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi} u(l) \\
&\quad + \sum_{l=k}^{\xi-1} \frac{k[T+1-l-\alpha(\xi-l)]}{T+1-\alpha\xi} u(l) + \sum_{l=\xi}^T \frac{k(T+1-l)}{T+1-\alpha\xi} u(l) \\
&= \sum_{l=1}^T g(k, l) u(l).
\end{aligned}$$

**Step 2.** We suppose  $k \geq \xi$ . Then the unique solution of problem (2.7), (2.1) can be written as

$$\begin{aligned}
y(k) &= - \left[ \sum_{l=1}^{\xi-1} (k-l)u(l) + \sum_{l=\xi}^{k-1} (k-l)u(l) \right] \\
&\quad + \frac{k}{T+1-\alpha\xi} \left[ \sum_{l=1}^{\xi-1} (T+1-l)u(l) + \sum_{l=\xi}^{k-1} (T+1-l)u(l) \right. \\
&\quad \left. + \sum_{l=k}^T (T+1-l)u(l) \right] - \frac{\alpha k}{T+1-\alpha\xi} \sum_{l=1}^{\xi-1} (\xi-l)u(l) \\
&= \sum_{l=1}^{\xi-1} \frac{l[T+1-k-\alpha(\xi-k)]}{T+1-\alpha\xi} u(l) \\
&\quad + \sum_{l=k}^{k-1} \frac{l(T+1-k) + \alpha\xi(k-l)}{T+1-\alpha\xi} u(l) + \sum_{l=k}^T \frac{k(T+1-l)}{T+1-\alpha\xi} u(l)
\end{aligned}$$



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$$= \sum_{l=1}^T g(k, l)u(l).$$

Thus the unique solution of problem (2.7), (2.1) can be written as  $y(k) = \sum_{l=1}^T g(k, l)u(l)$ .  $\square$

We observe that the condition  $\alpha\xi < T + 1$  implies that  $g(k, l)$  is nonnegative on  $\mathbf{N}_{T+1} \times \mathbf{N}_{1,T}$ , and positive on  $\mathbf{N}_{1,T} \times \mathbf{N}_{1,T}$ . From (2.3), we have

$$y(k) = \sum_{l=1}^T g(k, l)u(l),$$

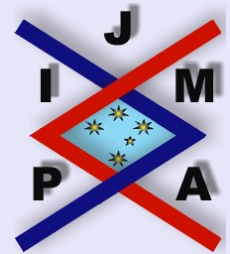
where

$$g(k, l) := (T + 1 - \alpha\xi)^{-1} (k(T + 1 - l) - (k - l)(T + 1 - \alpha\xi)\chi_{[1, k-1]}(l) - \alpha k(\xi - l)\chi_{[1, \xi-1]}(l)).$$

This is a positive function, which means that the finite set

$$\{g(k, l)/g(k, k) : k, l = 1, 2, \dots, T\}$$

takes positive values. Let  $M_1, M_2$  be its minimum and maximum values, respectively.



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### 3. Existence of Triple Solutions

In the following, we denote

$$m = \min_{k \in \mathbf{N}_{\xi, T}} \sum_{l=\xi}^T g(k, l), \quad M = \max_{k \in \mathbf{N}_{T+1}} \sum_{l=1}^T g(k, l)$$

and

$$\tilde{m} = \min_{k \in \mathbf{N}_{\xi, T}} g(k, k), \quad \tilde{M} = \max_{k \in \mathbf{N}_{T+1}} g(k, k).$$

Then  $0 < m < M, 0 < \tilde{m} < \tilde{M}$ .

Let  $E$  be the Banach space defined by

$$E = \{y : \mathbf{N}_{T+n-1} \longrightarrow \mathbb{R}, \Delta^i y(0) = 0, i = 0, 1, \dots, n-2\}.$$

Define

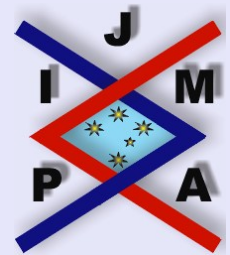
$$K = \left\{ y \in E : \Delta^{n-2} y(k) \geq 0 \text{ for } k \in \mathbf{N}_{T+1} \text{ and } \min_{k \in \mathbf{N}_{\xi, T}} \Delta^{n-2} y(k) \geq \sigma \|y\| \right\}$$

where  $\sigma = \frac{M_1 \tilde{m}}{M_2 \tilde{M}} \in (0, 1), \|y\| = \max_{k \in \mathbf{N}_{T+1}} |\Delta^{n-2} y(k)|$ . It is clear that  $K$  is a cone in  $E$ .

Finally, let the nonnegative continuous concave functional  $h : K \longrightarrow [0, \infty)$  be defined by

$$h(y) = \min_{k \in \mathbf{N}_{\xi, T}} \Delta^{n-2} y(k), \quad y \in K.$$

Note that for  $y \in K, h(y) \leq \|y\|$ .



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**Remark 1.** If  $y \in K$ ,  $\|y\| \leq c$ , then

$$0 \leq y(k) \leq qc, \quad k \in \mathbf{N}_{T+n-1},$$

where

$$q = q(n, T) = \frac{(T+n-1)(T+n) \cdots (T+2n-4)}{(n-2)!}.$$

In fact, if  $y \in K$ ,  $\|y\| \leq c$ , then  $0 \leq \Delta^{n-2}y(k) \leq c$ ,  $k \in \mathbf{N}_{T+1}$ , i.e.,

$$0 \leq \Delta(\Delta^{n-3}y(k)) = \Delta^{n-3}y(k+1) - \Delta^{n-3}y(k) \leq c.$$

Then one has

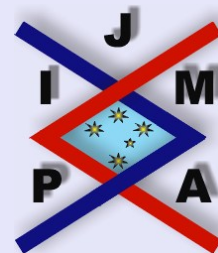
$$\begin{aligned} 0 &\leq \Delta^{n-3}y(1) - \Delta^{n-3}y(0) \leq c, \\ 0 &\leq \Delta^{n-3}y(2) - \Delta^{n-3}y(1) \leq c, \\ &\vdots \\ 0 &\leq \Delta^{n-3}y(k) - \Delta^{n-3}y(k-1) \leq c. \end{aligned}$$

We sum the above inequalities to obtain

$$0 \leq \Delta^{n-3}y(k) \leq kc, \quad k \in \mathbf{N}_{T+2}.$$

Similarly, we have

$$0 \leq \Delta^{n-4}y(k) \leq \left( \sum_{i=1}^k i \right) c = \frac{k(k+1)}{2!}c, \quad k \in \mathbf{N}_{T+3}.$$



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By using the induction method, one has

$$0 \leq y(k) \leq \frac{k(k+1) \cdots (k+n-3)}{(n-2)!} c, \quad k \in \mathbf{N}_{T+n-1}.$$

Then

$$0 \leq y(k) \leq \frac{(T+n-1)(T+n) \cdots (T+2n-4)}{(n-2)!} c = qc, \quad k \in \mathbf{N}_{T+n-1}.$$

**Theorem 3.1.** Assume that there exist constants  $a, b, c$  such that  $0 < a < b < c \cdot \min \left\{ \sigma, \frac{m}{M} \right\}$  and satisfy

$$(H_3) \quad f(k, y) \leq \frac{c}{M}, \quad (k, y) \in [0, T+n-1] \times [0, qc],$$

$$(H_4) \quad f(k, y) < \frac{a}{M}, \quad (k, y) \in [0, T+n-1] \times [0, qa],$$

$$(H_5) \quad \text{There exists some } l_0 \in [n-2, T+n-1], \text{ such that } f(k, y) \geq \frac{b}{m_0}, \quad (k, y) \in [n-2, T+n-1] \times [b, \frac{qb}{\sigma}], \text{ where } m_0 = \min_{k, l \in \mathbf{N}_T} g(k, l) > 0.$$

Then BVP (1.1) – (1.2) has at least three positive solutions  $y_1, y_2$  and  $y_3$ , such that

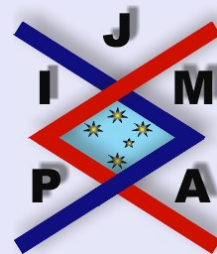
$$(3.1) \quad \|y_1\| < a, \quad h(y_2) > b,$$

and

$$(3.2) \quad \|y_3\| > a \quad \text{with} \quad h(y_3) < b.$$

*Proof.* Let the operator  $S : K \rightarrow E$  be defined by

$$(Sy)(k) = \sum_{l=1}^T G(k, l) f(l, y(l)), \quad k \in \mathbf{N}_{T+n-1}.$$



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It follows that

$$(3.3) \quad \Delta^{n-2}(Sy)(k) = \sum_{l=1}^T g(k, l) f(l, y(l)), \text{ for } k \in \mathbf{N}_{T+1}.$$

We shall now show that the operator  $S$  maps  $K$  into itself. For this, let  $y \in K$ , from  $(H_1)$ ,  $(H_2)$ , one has

$$(3.4) \quad \Delta^{n-2}(Sy)(k) = \sum_{l=1}^T g(k, l) f(l, y(l)) \geq 0, \text{ for } k \in \mathbf{N}_{T+1},$$

and

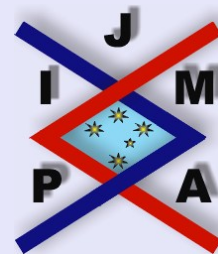
$$\begin{aligned} \Delta^{n-2}(Sy)(k) &= \sum_{l=1}^T g(k, l) f(l, y(l)) \\ &\leq M_2 \sum_{l=1}^T g(k, k) f(l, y(l)) \\ &\leq M_2 \widetilde{M} \sum_{l=1}^T f(l, y(l)), \quad \text{for } k \in \mathbf{N}_{T+1}. \end{aligned}$$

Thus

$$\|Sy\| \leq M_2 \widetilde{M} \sum_{l=1}^T f(l, y(l)).$$

From  $(H_1)$ ,  $(H_2)$ , and (3.3), for  $k \in \mathbf{N}_{\xi, T}$ , we have

$$\begin{aligned} \Delta^{n-2}(Sy)(k) &\geq M_1 \sum_{l=1}^T g(k, k) f(l, y(l)) \\ &\geq M_1 \widetilde{m} \sum_{l=1}^T f(l, y(l)) \geq \frac{M_1 \widetilde{m}}{M_2 \widetilde{M}} \|Sy\| = \sigma \|Sy\|. \end{aligned}$$



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Subsequently

$$(3.5) \quad \min_{k \in \mathbf{N}_{\xi, T}} \Delta^{n-2}(Sy)(k) \geq \sigma \|Sy\|.$$

From (3.4) and (3.5), we obtain  $Sy \in K$ . Hence  $S(K) \subseteq K$ . Also standard arguments yield that  $S : K \rightarrow K$  is completely continuous.

We now show that all of the conditions of Theorem 2.1 are fulfilled. For all  $y \in \overline{K_c}$ , we have  $\|y\| \leq c$ . From assumption  $(H_3)$ , we get

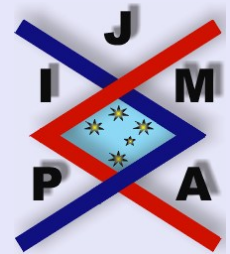
$$\begin{aligned} \|Sy\| &= \max_{k \in \mathbf{N}_{T+1}} |\Delta^{n-2}(Sy)(k)| \\ &= \max_{k \in \mathbf{N}_{T+1}} \left| \sum_{l=1}^T g(k, l) f(l, y(l)) \right| \\ &\leq \frac{c}{M} \max_{k \in \mathbf{N}_{T+1}} \sum_{l=1}^T g(k, l) = c. \end{aligned}$$

Hence  $S : \overline{K_c} \rightarrow \overline{K_c}$ .

Similarly, if  $y \in K_a$ , then assumption  $(H_4)$  yields  $f(k, y) < \frac{a}{M}$ , for  $k \in \mathbf{N}_{T+1}$ . As in the argument above, we can show  $S : \overline{K_a} \rightarrow K_a$ . Therefore, condition  $(A_2)$  of Theorem 2.1 is satisfied.

Now we prove that condition  $(A_1)$  of Theorem 2.1 holds. Let

$$y^*(k) = \frac{k(k+1) \cdots (k+n-3)b}{(n-2)! \sigma}, \quad \text{for } k \in \mathbf{N}_{\xi, T}.$$



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Then we can show that  $y^* \in K\left(h, b, \frac{qb}{\sigma}\right)$  and  $h(y^*) \geq \frac{b}{\sigma} > b$ . So

$$\left\{ y \in K\left(h, b, \frac{b}{\sigma}\right) : h(y) > b \right\} \neq \emptyset.$$

From assumptions  $(H_2)$  and  $(H_5)$ , one has

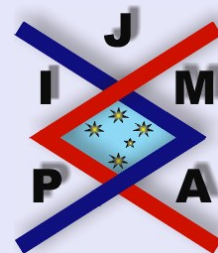
$$\begin{aligned} h(Sy) &= \min_{k \in \mathbf{N}_{\xi, T}} \sum_{l=1}^T g(k, l) f(l, y(l)) \\ &> \min_{k \in \mathbf{N}_{\xi, T}} \sum_{l=\xi}^T g(k, l) f(l, y(l)) \\ &\geq \min_{k \in \mathbf{N}_{\xi, T}} g(k, l_0) f(l_0, y(l_0)) \\ &\geq \frac{b}{m_0} \min_{k \in \mathbf{N}_{\xi, T}} g(k, l) \geq b. \end{aligned}$$

This shows that condition  $(A_1)$  of Theorem 2.1 is satisfied.

Finally, suppose that  $y \in K(h, b, c)$  with  $\|Sy\| > \frac{b}{\sigma}$ , then

$$h(Sy) = \min_{k \in \mathbf{N}_{\xi, T}} \Delta^{n-2}(Sy)(k) \geq \sigma \|Sy\| > b.$$

Thus, condition  $(A_3)$  of Theorem 2.1 is also satisfied. Therefore, Theorem 2.1 implies that BVP (1.1) – (1.2) has at least three positive solutions  $y_1, y_2, y_3$  described by (3.1) and (3.2).  $\square$




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**Corollary 3.2.** *Suppose that there exist constants*

$$0 < a_1 < b_1 < c_1 \cdot \min \left\{ \sigma, \frac{m}{M} \right\} < a_2 < b_2 < c_2 \cdot \min \left\{ \sigma, \frac{m}{M} \right\} < \dots < a_p,$$

*p is a positive integer, such that the following conditions are satisfied:*

$$(H_7) \quad f(k, y) < \frac{a_i}{M}, \quad (k, y) \in [0, T + n - 1] \times [0, qa_i], \quad i \in \mathbf{N}_{1,p};$$

$$(H_8) \quad \text{There exist } l_{i0} \in [n - 2, T + n - 1], \text{ such that } f(k, y) \geq \frac{qb_i}{m_0}, \quad (k, y) \in [n - 2, T + n - 1] \times [b_i, \frac{qb_i}{\sigma}], \quad i \in \mathbf{N}_{1,p-1}.$$

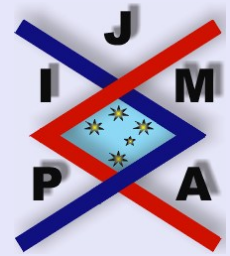
*Then BVP (1.1) – (1.2) has at least  $2p - 1$  positive solutions.*

*Proof.* When  $p = 1$ , from condition  $(H_7)$ , we show  $S : \overline{K_{a_1}} \longrightarrow K_{a_1} \subseteq \overline{K_{a_1}}$ . By using the Schauder fixed point theorem, we show that BVP (1.1) – (1.2) has at least one fixed point  $y_1 \in \overline{K_{a_1}}$ . When  $p = 2$ , it is clear that Theorem 3.1 holds (with  $c_1 = a_2$ ). Then we can obtain BVP (1.1) – (1.2) has at least three positive solutions  $y_1, y_2$  and  $y_3$ , such that  $\|y_1\| < a_1, h(y_2) > b_1, \|y_3\| > a_1$ , with  $h(y_3) < b_1$ . Following this way, we finish the proof by the induction method. The proof is completed.  $\square$

If the case  $n = 2$ , similar to the proof of Theorem 3.1, we obtain the following result.

**Corollary 3.3.** *Assume that there exist constants  $a, b, c$  such that  $0 < a < b < c \cdot \min \left\{ \sigma, \frac{m}{M} \right\}$  and satisfy*

$$(H_9) \quad f(k, y) \leq \frac{c}{M}, \quad (k, y) \in [0, T + n - 1] \times [0, c],$$



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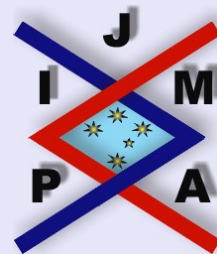


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$$(H_{10}) \quad f(k, y) < \frac{a}{M}, \quad (k, y) \in [0, T + n - 1] \times [0, a],$$

$$(H_{11}) \quad f(k, y) \geq \frac{b}{m}, \quad (k, y) \in [\xi, T + n - 1] \times [b, \frac{b}{\sigma}].$$

Then BVP (1.1) – (1.2) has at least three positive solutions  $y_1, y_2$  and  $y_3$ , satisfying (3.1) and (3.2).

Finally, we give an example to illustrate our main result.

**Example 3.1.** Consider the following second order third point boundary value problem

$$(3.6) \quad \Delta^2 y(k - 1) + f(k, y) = 0, \quad k \in \mathbf{N}_{1,6},$$

$$(3.7) \quad y(0) = 0, \quad y(7) = \frac{7}{9}y(3),$$

where  $f(k, y) = \frac{100}{k+100}a(y)$ , and

$$a(y) = \begin{cases} \frac{1}{720} + \sin^8 y, & \text{if } y \in [0, \frac{1}{30}]; \\ \frac{1}{720} + 6(y - \frac{1}{30}) + \sin^8 y, & \text{if } y \in [\frac{1}{30}, 3]; \\ \frac{1}{720} + \frac{89}{5} + \frac{\sin^2(y-3)}{2} + \sin^8 y, & \text{if } y \in [3, 360]. \end{cases}$$

Then  $T = 6, n = 3, \alpha = \frac{7}{9} < 1, T + 1 - \alpha n = \frac{14}{3} > 0$ . Then the conditions

$(H_1), (H_2)$  are satisfied, and the function

$$G(k, l) = \frac{3}{14} \begin{cases} \frac{l(42-2k)}{9}, & l \in \mathbb{N}_{1,k-1} \cap \mathbb{N}_{1,2}; \\ \frac{3l(7-k)+7(k-l)}{3}, & l \in \mathbb{N}_{3,k-1}; \\ \frac{k(42-2l)}{9}, & l \in \mathbb{N}_{k,2}; \\ k(7-l), & l \in \mathbb{N}_{k,6} \cap \mathbb{N}_{3,6}, \end{cases}$$

is the Green's function of the problem  $-\Delta^2 y(k-1) = 0, k \in \mathbb{N}_{1,6}$  with (3.7).

Thus we can compute  $m = \frac{27}{2}, M = 18, \tilde{m} = \frac{9}{7}, \tilde{M} = \frac{18}{7}, M_1 = \frac{2}{9}, M_2 = 9,$   
 $\sigma = \frac{1}{81} < \frac{m}{M} = \frac{3}{4}.$  We choose that  $a = \frac{1}{35}, b = \frac{1}{10}, c = 360,$  consequently,

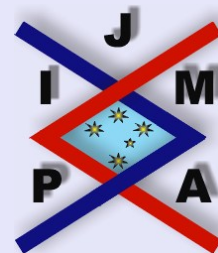
$$f(k, y) = \frac{100}{k+100} a(y) \leq a(y) \leq \begin{cases} \frac{1}{720} + 1 < 20 = \frac{c}{M}, & (k, y) \in [0, 7] \times [0, \frac{1}{30}]; \\ \frac{1}{720} + 6(3 - \frac{1}{30}) + 1 < 20 = \frac{c}{M}, & (k, y) \in [0, 7] \times [\frac{1}{30}, 3]; \\ \frac{1}{720} + \frac{89}{5} + \frac{3}{2} < 20 = \frac{c}{M}, & (k, y) \in [0, 7] \times [3, 360]. \end{cases}$$

Thus

$$f(k, y) \leq \frac{c}{M}, \quad (k, y) \in [0, 7] \times [0, 360];$$

and

$$f(k, y) \leq \frac{1}{720} + \sin^8 y < \frac{1}{630} = \frac{a}{M}, \quad (k, y) \in [0, 7] \times \left[0, \frac{1}{35}\right];$$



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$$f(k, y) \geq \frac{100}{107} \left[ \frac{1}{720} + 6 \left( \frac{1}{10} - \frac{1}{30} \right) + \sin^8 y \right] \geq \frac{1}{135} = \frac{b}{m},$$

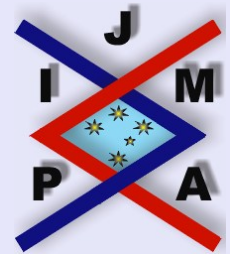
$$(k, y) \in [3, 7] \times \left[ \frac{1}{27}, 3 \right].$$

That is to say, all the conditions of Corollary 3.3 are satisfied. Then the boundary value problem (3.6), (3.7) has at least three positive solutions  $y_1, y_2$  and  $y_3$ , such that

$$y_1(k) < \frac{1}{35}, \text{ for } k \in \mathbf{N}_7, \quad y_2(k) > \frac{1}{27}, \text{ for } k \in \mathbf{N}_{3,7},$$

and

$$\max_{k \in \mathbf{N}_{1,7}} y_3(k) > \frac{1}{35}, \text{ for } k \in \mathbf{N}_7 \text{ with } \min_{k \in \mathbf{N}_{3,7}} y_3(k) < \frac{1}{27}.$$




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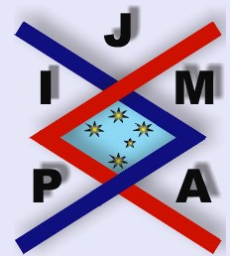
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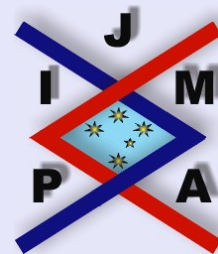
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