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REFINING SOME INEQUALITIES

RICĂ ZAMFIR
BD IULIU MANIU 55A BLOC 17A, AP. 5 SECT. 6
BUCURESTI, ROMANIA
rzamfir62@gmail.com

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ABSTRACT. In this article we improve two well known bounds for the roots of polynomials with complex coefficients. Our method is algebraic, unitary and was used among others by L. Panaitopol and D. Stefănescu.

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1. INTRODUCTION

Determining bounds for the zeros of polynomials is a classical problem to which many authors have made contributions, beginning with Gauss and Cauchy. Since the days of Gauss and Cauchy many other mathematicians have contributed to the further growth of the subject, using various methods (the theory of analytical functions, matrix analysis, the theory of operators ,differential equations of second order).

In [6] Williams established the following result:

Theorem 1.1. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 + a_0 \in \mathbf{C}[X]$, $a_n \neq 0$ and z is an arbitrary root of f, then:*

$$|z|^2 \leq 1 + \left|\frac{a_0}{a_n}\right|^2 + \left|\frac{a_1 - a_0}{a_n}\right|^2 + \dots + \left|\frac{a_n - a_{n-1}}{a_n}\right|^2. \tag{1.1}$$

In [1, p. 151] we find a statement that can be reformulated as:

Proposition 1.2. *If f is polynomial like in Theorem 1.1 and $p \in \{1, 2, \dots, n\}$, then at least p roots of f are within the disk:*

$$|z| \leq 1 + \left(\sum_{j=0}^{p-1} \left|\frac{a_j}{a_n}\right|^2\right)^{\frac{1}{2}}. \tag{1.2}$$

In what follows we want to refine the inequalities (1.1) and (1.2), by applying a unitary method, used by L.Panaitopol and D. Stefănescu.

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2. THE MAIN RESULTS

In this section we present Theorems 2.1 and 2.2 which establish refinements of inequalities (1.1) and (1.2).

Theorem 2.1. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \in \mathbf{C}[X]$ let $b_0 = a_0$, $b_1 = a_1 - a_0, \ldots,$ $b_n = a_n - a_{n-1}$. Then, for any root z of f, we have:*

$$|z|^2 \leq 1 + \sum_{j=0}^{n} \left| \frac{b_j}{a_n} \right|^2 - \frac{(\mathrm{Re}(b_0\overline{b}_1 + b_1\overline{b}_2 + \cdots + b_{n-1}\overline{b}_n - b_n\overline{a}_n))^2}{(|b_0|^2 + |b_1|^2 + \cdots + |b_n|^2) \cdot |a_n|^2}. \tag{2.1}$$

Remark 1. If $b_0\overline{b}1 + b_1\overline{b}_2 + \cdots + b_{n-1}\overline{b}_n - b_n\overline{a}_n \neq 0$, then inequality (2.1) is better than inequality (1.1).

Theorem 2.2. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \in \mathbf{C}[X]$ and $p \in \{1, 2, ..., n\}$, then at least p roots of f are within the disk:*

$$|z| \leq 1 + \left(\sum_{j=0}^{p-1} \left| \frac{a_j}{a_n} \right|^2 - \frac{(Re(a_0\overline{a}_1 + a_1\overline{a}_2 + \cdots + a_{p-1}\overline{a}_p))^2}{(|a_0|^2 + \cdots + |a_p|^2) \cdot |a_n|^2} \right)^{\frac{1}{2}}. \tag{2.2}$$

3. PROOFS OF MAIN THEOREMS

Proof of Theorem 2.1. We consider the polynomial

$$F(x) = (x - \alpha)f(x),$$

where α is a real number. The coefficients of polynomial F are:

$$c_k = a_{k-1} - \alpha a_k,$$

where $k = \overline{0, n+1}$ and $a_{-1} = a_{n+1} = 0$. By applying Theorem 1.1 to polynomial F, we find that if z is a root of F then:

$$|z|^2 \leq 1 + \left| \frac{c_0}{c_{n+1}} \right|^2 + \left| \frac{c_1 - c_0}{c_{n+1}} \right|^2 + \cdots + \left| \frac{c_{n+1} - c_n}{c_{n+1}} \right|^2. \tag{3.1}$$

We compute and obtain:

$$1 + \left| \frac{c_0}{c_{n+1}} \right|^2 + \sum_{k=0}^{n} \left| \frac{c_{k+1} - c_k}{c_{n+1}} \right|^2 = 1 + \alpha^2 \left| \frac{b_0}{a_n} \right|^2 + \sum_{k=0}^{n} \left| \frac{b_k - \alpha b_{k+1}}{a_n} \right|^2 + \left| \frac{b_n + \alpha a_n}{a_n} \right|^2.$$

Further, we have:

$$\begin{aligned} |b_k - \alpha b_{k+1}|^2 &= (b_k - \alpha b_{k+1})(\overline{b_k - \alpha b_{k+1}}) \\ &= (b_k - \alpha b_{k+1})(\overline{b}_k - \alpha \overline{b}_{k+1}) \\ &= |b_k|^2 + \alpha^2 |b_{k+1}|^2 - 2\alpha \,\mathrm{Re}(b_k \overline{b}_{k+1}) \end{aligned}$$

and therefore, if we use the notation:

$$\begin{aligned} A &= |b_0|^2 + |b_1|^2 + \cdots + |b_n|^2 \\ B &= \mathrm{Re}(b_0\overline{b}_1 + b_1\overline{b}_2 + \cdots + b_{n-1}\overline{b}_n - b_n\overline{a}_n), \end{aligned}$$

then:

$$1 + \left| \frac{c_0}{c_{n+1}} \right|^2 + \sum_{k=0}^{n} \left| \frac{c_{k+1} - c_k}{c_{n+1}} \right|^2 = 1 + \frac{1}{|a_n|^2}(A\alpha^2 - 2B\alpha + A). \tag{3.2}$$

Using inequality (3.1) and relation (3.2) we obtain that for any roots of F we have:

$$|z|^2 \leq 1 + g(\alpha), \tag{3.3}$$

where

$$g(\alpha) = \frac{1}{|a_n|^2}(A\alpha^2 + 2B\alpha + A). \tag{3.4}$$

It is clear that g is minimal for $\alpha = \frac{B}{A}$ and the minimal value is:

$$g_{min} = \frac{1}{|a_n|^2} \cdot \left(A - \frac{B^2}{A}\right). \tag{3.5}$$

From (3.3) and (3.5) we obtain that

$$|z|^2 \leq 1 + \frac{A}{|a_n|^2} - \frac{B^2}{A \cdot |a_n|^2} \tag{3.6}$$

which takes place for any root z of F, and therefore for any root of f, which concludes the proof. □

Proof of Theorem 2.2. As in the demonstration of Theorem 2.1, we consider the polynomial

$$F_\alpha(x) = (x - \alpha) f(x).$$

If we apply Proposition 1.2 to F_α, we will find that at least p roots of F_α are located inside the disk:

$$|z| \leq 1 + \left(\sum_{j=0}^{p-1} \left|\frac{c_j}{c_{n+1}}\right|^2\right)^{\frac{1}{2}}. \tag{3.7}$$

We have:

$$\begin{aligned}
\sum_{j=0}^{p-1} \left|\frac{c_j}{c_{n+1}}\right|^{\frac{1}{2}} &= \sum_{j=0}^{p-1} \frac{|a_{j-1}|^2 + \alpha^2 |a_j|^2 - 2\alpha \operatorname{Re}(a_{j-1} \cdot \overline{a}_j))}{|a_n|^2} \\
&= \frac{1}{|a_n|^2} \cdot (A_1\alpha^2 - 2B_1\alpha + C_1) \\
&= h(\alpha),
\end{aligned}$$

where we used the following notations:

$$\begin{aligned}
A_1 &= |a_0|^2 + |a_1|^2 + \cdots + |a_{p-1}|^2 \\
B_1 &= \operatorname{Re}(a_0 \cdot \overline{a}_1 + a_1 \cdots \overline{a}_2 + \cdots + a_{p-2} \cdot \overline{a}_{p-1}) \\
C_1 &= |a_0|^2 + |a_1|^2 + \cdots + |a_{p-2}|^2.
\end{aligned}$$

The minimal value of h is obtained for

$$\alpha_1 = \frac{B_1}{A_1} \tag{3.8}$$

and it is:

$$h_{min} = \frac{1}{|a_n|^2} \cdot \left(C_1 - \frac{B_1^2}{A_1}\right). \tag{3.9}$$

From (3.7) we deduce that inside the disk

$$|z| \leq 1 + \frac{1}{|a_n|} \cdot \left(C_1 - \frac{B_1^2}{A_1}\right)^{\frac{1}{2}} \tag{3.10}$$

there are at least p roots of F_α.

We apply this result for the polynomial F_{α_1} where α_1 is given by (3.8) and we obtain that the polynomial F_{α_1} has at least p roots inside the disk given by (3.10).

Since α_1 verifies the inequality (3.10) (a simple calculation shows that we have $|\alpha| \le 1$), one of these p roots is α_1 and the other $p-1$ roots of F_{α_1} inside the disk (3.10) are actually roots of f.

We have therefore proved that at least $p-1$ roots of f are inside the disk

$$|z| \le 1 + \left(\sum_{j=0}^{p-2} \left| \frac{a_j}{a_n} \right|^2 - \frac{(\mathrm{Re}(a_0\overline{a}_1 + a_1\overline{a}_2 + \cdots + a_{p-2}\overline{a}_{p-1}))^2}{(|a_0|^2 + \cdots + |a_{p-1}|^2) \cdot |a_n|^2} \right)^{\frac{1}{2}} \tag{3.11}$$

and, as a result, there are at least p roots of f inside the disk

$$|z| \le 1 + \left(\sum_{j=0}^{p-1} \left| \frac{a_j}{a_n} \right|^2 - \frac{(\mathrm{Re}(a_0\overline{a}_1 + a_1\overline{a}_2 + \cdots + a_{p-1}\overline{a}_{p}))^2}{(|a_0|^2 + \cdots + |a_{p}|^2) \cdot |a_n|^2} \right)^{\frac{1}{2}}, \tag{3.12}$$

which concludes the proof. □

Corollary 3.1. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \in \mathbf{C}[X]$, $a_n \neq 0$, then all the roots of f are inside the disk:*

$$|z| \le 1 + \left(\sum_{j=0}^{n-1} \left| \frac{a_j}{a_n} \right|^2 - \frac{(\mathrm{Re}(a_0\overline{a}_1 + a_1\overline{a}_2 + \cdots + a_{n-1}\overline{a}_{n}))^2}{(|a_0|^2 + \cdots + |a_{n}|^2) \cdot |a_n|^2} \right)^{\frac{1}{2}}. \tag{3.13}$$

Proof. We apply Theorem 2.2 for $p = n$. □

Corollary 3.2. *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \in \mathbf{C}[X]$, $a_0 \neq 0$ and*

$$M^2 = \sum_{j=0}^{n-p-1} \left| \frac{a_{n-j}}{a_0} \right|^2 - \frac{(Re(a_{p-1}\overline{a}_p + a_p\overline{a}_{p+1} + \cdots + a_{n-1}\overline{a}_{n}))^2}{(|a_p|^2 + |a_{p+1}|^2 + \cdots + |a_{n}|^2) \cdot |a_0|^2}, \tag{3.14}$$

then f has at most p roots inside the disk

$$|z| \le \frac{1}{1+M}. \tag{3.15}$$

Proof. We apply Theorem 2.2 to the reciprocal polynomial $f^*(x) = x^n f(\frac{1}{x})$. □

4. APPLICATIONS

(1) Let $f(x) = 20x^4 - 2x^3 + 2x^2 - x + 1$. Using the Mathematica program we can find the roots of f:

$$\begin{aligned} z_1 &= -0.271695 - 0.417344i, \\ z_2 &= -0.271695 + 0.417344i, \\ z_3 &= 0.321695 - 0.313257i, \\ z_4 &= 0.321695 + 0.313257i. \end{aligned}$$

It is clear that for every root z we have $|z| < 1$. Appliyng the theorem of Williams, we find $|z| < 1.5116$. If we apply Theorem 2.1 we find a better bound:

$$|z| < 0.907$$

(2) Let $f(x) = 6x^4 + 35x^3 + 31x^2 + 35x + 6$. If we apply Theorem 1.1 we find:

$$|z| \leq 7.043,$$

and if we apply Theorem 2.1 we find:

$$|z| \leq 7.032.$$

Note that the root of maximal modulus is $z = -5.028$.

(3) Let $f(x) = 7x^5 - 20x^3 + x + 1$. Appliyng Theorem 2.1 we find that every root z of f is inside the disk:

$$|z| \leq 4.048$$

while Theorem 1.1 gives:

$$|z| \leq 4.288$$

(4) Let $f(x) = 10x^5 + x^4 + 100x^3 + 10x^2 + 90x + 1$. If we apply Theorem 1.1 we find:

$$|z| \leq 18.001$$

and if we apply Theorem 2.1 we find:

$$|z| \leq 12.529.$$

(5) Let$f(x) = x^5 + 7x^4 + 55x^3 + 112x^2 + x + 1$. Applying Theorem 2.2 for $p = 1$, we obtain that f has at least one root inside the disk

$$D = \{z \in \mathbf{C}; |z| \leq 1.707\}.$$

The roots of f are:

$$z_1 = -2.561, \quad z_2 = -2.216 + 6.219i, \quad z_3 = \overline{z}_2,$$
$$z_4 = -0.002 + 0.094i, \quad z_5 = \overline{z}_4$$

and we see that $z_4, z_5 \in D$.

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