Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 5, Article 138, 2005

π AND SOME OTHER CONSTANTS

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Received 18 March, 2005; accepted 26 October, 2005
Communicated by J. Sándor

ABSTRACT. We consider a particular definite integral and reduce it to hypergeometric form. Then we develop identities for some numerical constants and the number π.

Key words and phrases: Hypergeometric summation, Definite integration, Binomial type series.

2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 33C05.

1. INTRODUCTION

π, is a real number and defined as the ratio of the circumference of a circle to its diameter. π'^s digits have many interesting properties and π has a rich history dating back to the time of the Babylonians and Egyptians, circa 2000 B.C. The Bible has two references, I Kings 7:23 and Chronicles 4:2, to Pi and gives it an estimate of about 3. The Babylonians gave an estimate of π as $3\frac{1}{8}$ and the Egyptians also obtained $3\frac{13}{81}$ as an estimate. We know that $3\frac{1}{8} < \pi < 3\frac{13}{81}$.

Many researchers have increasingly calculated the number of decimal places for the value of π. Apparently in September 2002, Dr. Kanada and his team, from the University of Tokyo, calculated π to 1.2411 trillion digits, indeed a world record. Many, many formulae also exist for the representation of π, and a collection of these formulae is listed below.

Vieta (~1593), see [7], gave an infinite product of nested radicals for the reciprocal of π, namely

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots .$$

ISSN (electronic): 1443-5756

This paper is based on the talk given by the author within the "International Conference of Mathematical Inequalities and their Applications, I", December 06-08, 2004, Victoria University, Melbourne, Australia [http://rgmia.vu.edu.au/conference].

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Wallis (~1650), see [7], gave

$$\frac{\pi}{4}=\prod_{r=1}^{\infty}\left(1-\frac{1}{(2r+1)^2}\right).$$

Leibnitz (~1670), see [7], gave the very slow converging series

$$\frac{\pi}{4}=\sum_{r=0}^{\infty}\frac{(-1)^r}{2r+1}.$$

Newton (~1666) admitted being ashamed at having computed π to fifteen decimal places, by the formula

$$\begin{aligned}\pi &= \frac{3\sqrt{3}}{4}+24\int_0^{\frac{1}{4}}\sqrt{x-x^2}dx\\ &= \frac{3\sqrt{3}}{4}+2-\frac{3}{4}\left\{\frac{1}{5}+\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^7}+\frac{5}{11\cdot 2^{12}}+\cdots\right\}\\ &= \frac{3\sqrt{3}}{4}+2-\frac{3}{4}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{16^k(k+1)(2k+5)}\end{aligned}$$

Euler (~1750) gave many representations of π including:

$$\frac{\pi}{2}=\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{6n^2}+4n\sum_{j=1}^{n}\frac{1}{n^2+j^2}\right].$$

Ramanujan (~1914) has also given many representations of π and its reciprocal, including:

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{3^4\cdot 11^2}\sum_{r=0}^{\infty}\frac{(4r)!\,(1103+(2\cdot 5\cdot 7\cdot 13\cdot 29)\,r)}{(r!)^4\,(2^2\cdot 3^2\cdot 11)^{4r}}.$$

For a fuller account of Ramanujan's work the interested reader is referred to the books of Berndt [4].

Comtet (1974) gave

$$\pi^4=\frac{2^3\cdot 3^4\cdot 5}{17}\sum_{r=1}^{\infty}\frac{1}{r^4\binom{2r}{r}}.$$

D. and G. Chudnovsky (1989) gave

$$\frac{1}{\pi}=12\sum_{r=0}^{\infty}\frac{(-1)^r\,(6r)!}{(r!)^3\,(3r)!}\cdot\frac{13\cdot 1045493+2\cdot 3^2\cdot 7\cdot 11\cdot 19\cdot 127\cdot 163r}{(2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3)^{r+\frac{1}{2}}}.$$

Bailey, Borwein and Plouffe (1996) gave

$$\pi=\sum_{r=0}^{\infty}\frac{1}{16^r}\left[\frac{4}{8r+1}-\frac{2}{8r+4}-\frac{1}{8r+5}-\frac{1}{8r+6}\right]. \tag{1.1}$$

Bellard (1997) gave

$$\pi=\frac{1}{3^2\cdot 5^2\cdot 11\cdot 13\cdot 23}\left[\sum_{r=1}^{\infty}\frac{3P(r)}{\binom{7r}{2r}2^{r-1}}-2^4\cdot 5\cdot 254741\right],$$

where

$$\begin{aligned}P(r)&=-13\cdot 29\cdot 2351653r^5+193\cdot 16193509r^4-5^2\cdot 7\cdot 79\cdot 212873r^3\\ &\quad+5\cdot 206392559r^2-2\cdot 98441137r+2^3\cdot 13\cdot 43\cdot 2459.\end{aligned}$$

Lupas [10], gave

$$\pi = 4 + \sum_{k=1}^{\infty} (-16)^k \frac{\binom{2k}{k}\left(40k^2+16k+1\right)}{\binom{4k}{2k}^2 2k\left(4k+1\right)^2}.$$

The original Lupas formula contained a minor misprint which has been corrected here.

Borwein and Girgensohn (2003), wrote

$$\pi = \ln 4 + 10 \sum_{r=1}^{\infty} \frac{1}{2^r r \binom{3r}{r}}.$$

Sofo [16] has given

$$\frac{\pi}{2} = \sqrt{2} + \ln\left(\sqrt{2}-1\right) + \sum_{r=0}^{\infty} \binom{4r}{2r} \frac{1}{16^r \left(2r+1\right)\left(4r+1\right)}$$

and

$$\pi^2 = \frac{1308}{135} + \frac{12}{5} \sum_{r=1}^{\infty} \frac{4^r}{r^2 \binom{2r}{r}\left(r+1\right)\left(2r+1\right)\left(2r+3\right)}.$$

Many other representations of π exist, including the famous Machin-type formulae such as

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

There are also many other connections of π and other mathematical constants, including:

$$e^{i\pi} + 1 = 0,$$

$$\pi^3 + 8\pi = 56 - 8 \sum_{r=1}^{\infty} \frac{(-1)^r}{r\left(r+1\right)\left(2r+1\right)^3},$$

$$\frac{\pi^2}{6} = 3\ln^2 \phi + \sum_{r=0}^{\infty} \frac{(-1)^r (r!)^2}{(2r)!\left(2r+1\right)^2}, \text{ where } \phi \text{ is the Golden ratio}$$

and

$$\pi^2 = -12e^3 \sum_{r=1}^{\infty} \frac{1}{r^2} \cos\left(\frac{9}{r\pi + \sqrt{\left(r\pi\right)^2 - 3^2}}\right).$$

A selection of some series expansion representations of π including some of the above is given by Sebah and Gourdon [15].

There are other nice articles and books relating to π including [2, 3, 6, 7, 8, 11, 12].

The aim of this paper is to derive representations of π, as well as some other constants, by the consideration of a particular definite integral. The following integral will now be investigated.

2. THE INTEGRAL

Theorem 2.1. *For k, m and α real positive numbers and $a \geq 1$, then*

$$I(a,k,m,\alpha) = \int_0^{\frac{1}{a}} \frac{x^m}{(1-x^k)^\alpha} dx \tag{2.1}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!\,(rk+m+1)\,a^{rk+m+1}} \tag{2.2}$$

$$= T_0 \; {}_2F_1\left[\begin{array}{c} \frac{m+1}{k}, \; \alpha \\ \frac{m+1+k}{k} \end{array}\middle|\, \frac{1}{a^k}\right] \tag{2.3}$$

$$= \frac{1}{k}\left[B\left(1-\alpha, \frac{m+1}{k}\right) - B\left(1-a^{-k}; 1-\alpha, \frac{m+1}{k}\right)\right], \tag{2.4}$$

where

$$T_0 = \frac{1}{(m+1)\,a^{m+1}}, \tag{2.5}$$

$(b)_s$ *is Pochhammer's symbol defined by*

$$\begin{cases} (b)_0 = 1 \\ \\ (b)_s = b(b+1)\cdots(b+s-1) = \frac{\Gamma(b+s)}{\Gamma(b)}, \end{cases} \tag{2.6}$$

$\Gamma(b)$ *is the classical Gamma function,* ${}_2F_1[\cdot\,\cdot]$ *is the Gauss Hypergeometric function,* $B(s,t)$ *is the classical Beta function and*

$$B(z;s,t) = \int_0^z u^{s-1}(1-u)^{t-1}\,du$$

is the incomplete Beta function.

Proof.

$$\begin{aligned} I(a,k,m,\alpha) &= \int_0^{\frac{1}{a}} \frac{x^m}{(1-x^k)^\alpha} dx \\ &= \int_0^{\frac{1}{a}} \sum_{r=0}^{\infty} (-1)^r \binom{-\alpha}{r} x^{kr+m}. \end{aligned}$$

where we have utilised

$$\frac{1}{(1+z)^\beta} = \sum_{r=0}^{\infty} \binom{-\beta}{r} z^r$$

and from

$$\binom{-\beta}{r} = (-1)^r \binom{\beta+r-1}{r} = \frac{(-1)^r (\beta)_r}{r!}$$

we have

$$I(a,k,m,\alpha) = \int_0^{\frac{1}{a}} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} x^{kr+m}.$$

Reversing the order of integration and summation and substituting the integration limits we obtain the result (2.2). The result (2.4) is obtained by the use of the substitution $u = 1 - x^k$.

Binomial sums are intrinsically associated with generalised hypergeometric functions and if from (2.2) we let

$$T_r = \frac{(\alpha)_r}{r!\,(rk+m+1)\,a^{rk+m+1}}, \tag{2.7}$$

then we get the ratio

$$\frac{T_{r+1}}{T_r} = \frac{(\alpha+r)\left(r+\frac{m+1}{k}\right)}{a^k\,(r+1)\left(r+\frac{m+1+k}{k}\right)} \tag{2.8}$$

where T_0 is given by (2.5). From (2.5) and (2.2) we can write

$$\begin{aligned} I(a,k,m,\alpha) &= T_0\;{}_2F_1\left[\left.\begin{matrix}\frac{m+1}{k},\ \alpha\\ \frac{m+1+k}{k}\end{matrix}\ \right|\ \frac{1}{a^k}\right] \\ &= \frac{1}{k}\left[B\left(1-\alpha,\frac{m+1}{k}\right) - B\left(1-a^{-k};1-\alpha,\frac{m+1}{k}\right)\right], \end{aligned}$$

which is the result (2.3). We can now match (2.2) and (2.3) so that

$$\begin{aligned} \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!\,(rk+m+1)\,a^{rk+m+1}} &= T_0\;{}_2F_1\left[\left.\begin{matrix}\frac{m+1}{k},\ \alpha\\ \frac{m+1+k}{k}\end{matrix}\ \right|\ \frac{1}{a^k}\right] \\ &= \frac{1}{k}\left[B\left(1-\alpha,\frac{m+1}{k}\right) - B\left(1-a^{-k};1-\alpha,\frac{m+1}{k}\right)\right], \end{aligned}$$

and the infinite series converges for $\left|a^{-k}\right| < 1$. □

In a previous paper, Sofo [17] has utilised (2.1) for the case $a = 1$ and developed identities for π and other constants, such as

$$\pi = \frac{15p!\left(\sqrt{30+6\sqrt{5}}+1-\sqrt{5}\right)}{4\left(\frac{7}{30}\right)_p}\sum_{r=0}^{\infty}\frac{\left(\frac{7}{30}\right)_r}{r!\,(30r+30p+7)}, \qquad \text{for } p = 0,1,2,3,\ldots$$

Remark 2.2. Bailey, Borwein, Borwein and Plouffe see [7] utilised (2.1) for $a = \sqrt{2}$, $\alpha = 1$, $k = 8$ and $m = \beta - 1$, $\beta < 8$ to prove the new formula (1.1). Subsequently Hirschhorn [9] has shown that (1.1) can be obtained from standard integration procedures.

The following lemma will be useful in the consideration of the integral (2.1).

Lemma 2.3. *For* $p = 0,1,2,\ldots$ *the following well-known identities are given by Beyer* [5]

$$\sin^{2p+1} x = \frac{(-1)^p}{2^{2p}}\sum_{j=0}^{p}(-1)^j\binom{2p+1}{p}\sin\left((2p+1-2j)\,x\right) \tag{2.9}$$

and

$$\begin{aligned} &\sin^{2p+2} x \\ &\quad = \frac{1}{2^{2p+1}}\left[\binom{2p+1}{p} - (-1)^p\sum_{j=0}^{p}(-1)^j\binom{2p+2}{p}\cos\left((2p+2-2j)\,x\right)\right]. \end{aligned} \tag{2.10}$$

The integral (2.1) can be simplified as follows. Consider the case $k = 2$; then from (2.1)

$$I\,(a,m,2,\alpha) = \int_0^{\frac{1}{a}}\frac{x^m}{(1-x^2)^\alpha}dx. \tag{2.11}$$

The following lemma concerns the integral (2.11).

Lemma 2.4.

(i) *For* $m = 2p+1$, $2\alpha = 2q+1$; $p = 0, 1, 2, \ldots$, *and* $q = 0, 1, 2, \ldots$

$$I(a,m,2,\alpha) = \int_0^{\frac{1}{a}} \frac{x^{2p+1}}{(1-x^2)^{\frac{2q+1}{2}}} dx = \int_0^{\theta^*} \frac{\sin^{2p+1}\theta}{\cos^{2q}\theta} d\theta \tag{2.12}$$
$$= \frac{1}{2}\left[B\left(\frac{1-2q}{2}, p+1\right) - B\left(1-a^{-2}; \frac{1-2q}{2}, p+1\right)\right]$$
$$= \frac{1}{2q-1} \cdot \frac{\left(\frac{1}{a}\right)^{2p+2}}{\cos^{2q-1}\theta^*} + \sum_{j=1}^{q-1} \frac{(-1)^j \left(\frac{1}{a}\right)^{2p+2}}{\cos^{2(q-j)-1}\theta^*} \prod_{i=1}^{j} \frac{2(p-q+i)-1}{2(q-i)+1}$$
$$+ (-1)^q \prod_{i=1}^{q} \frac{2(p-q+i)-1}{2(q-i)+1} \left[\frac{(-1)^p}{2^{2p}}\right.$$
$$\left. \times \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+1} \binom{2p+1}{s} \{1 - \cos((2p-2s+1)\theta^*)\}\right].$$

where $x = \sin\theta$ *and* $\theta^* = \arcsin\left(\frac{1}{a}\right)$.
Note, the middle term in the right hand side of (2.12) is identically zero for $q = 1$ *and only the last sum applies for* $q = 0$.

(ii) *For* $m = 2p+2$, $2\alpha = 2q+1$; $p = 0, 1, 2, \ldots$, *and* $q = 0, 1, 2, \ldots$

$$I(a,m,2,\alpha) = \int_0^{\frac{1}{a}} \frac{x^{2p+2}}{(1-x^2)^{\frac{2q+1}{2}}} dx = \int_0^{\theta^*} \frac{\sin^{2p+2}\theta}{\cos^{2q}\theta} d\theta \tag{2.13}$$
$$= \frac{1}{2}\left[B\left(\frac{1-2q}{2}, p+\frac{3}{2}\right) - B\left(1-a^{-2}; \frac{1-2q}{2}, p+\frac{3}{2}\right)\right]$$
$$= \frac{1}{2q-1} \cdot \frac{\left(\frac{1}{a}\right)^{2p+3}}{\cos^{2q-1}\theta^*} + \sum_{j=1}^{q-1} \frac{(-1)^j \left(\frac{1}{a}\right)^{2p+3}}{\cos^{2(q-j)-1}\theta^*} \prod_{i=1}^{j} \frac{2(p-q+i+1)}{2(q-i)+1}$$
$$+ (-1)^q \prod_{i=1}^{q} \frac{2(p-q+i+1)}{2(q-i)+1} \left\{\frac{1}{2^{2p+1}}\left[\binom{2p+1}{p}\theta^*\right.\right.$$
$$\left.\left. - (-1)^p \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+2} \binom{2p+2}{s} \sin((2p-2s+2)\theta^*)\right]\right\}.$$

where $x = \sin\theta$ *and* $\theta^* = \arcsin\left(\frac{1}{a}\right)$.
Note, the middle term in the right hand side of (2.13) is identically zero for $q = 1$ *and only the last two terms apply for* $q = 0$.

Proof. (i) From

$$I(a,m,2,\alpha) = \int_0^{\theta^*} \frac{\sin^{2p+1}\theta}{\cos^{2q}\theta} d\theta,$$

integrating by parts once leads to

$$I(a,m,2,\alpha) = \frac{1}{2q-1}\left[\left.\frac{\sin^{2p+2}\theta}{\cos^{2q-1}\theta}\right\}_0^{\theta^*} - (2p+3-2q)\int_0^{\theta^*} \frac{\sin^{2p+1}\theta}{\cos^{2q-2}\theta} d\theta\right]$$
$$= \frac{\left(\frac{1}{a}\right)^{2p+2}}{(2q-1)\cos^{2q-1}\theta^*} - \frac{2p-2q+3}{2q-1}\int_0^{\theta^*} \frac{\sin^{2p+1}\theta}{\cos^{2q-2}\theta} d\theta,$$

and repeated integration by parts gives us

$$I\left(a,m,2,\alpha\right) \tag{2.14}$$
$$= \frac{\left(\frac{1}{a}\right)^{2p+2}}{\left(2q-1\right)\cos^{2q-1}\theta^*} + \sum_{j=1}^{q-1} \frac{\left(-1\right)^j \left(\frac{1}{a}\right)^{2p+2}}{\cos^{2(q-j)-1}\theta^*} \prod_{i=1}^{j} \frac{2\left(p-q+i\right)+1}{2\left(q-i\right)+1}$$
$$+ \left(-1\right)^q \prod_{i=1}^{q} \frac{2\left(p-q+i\right)+1}{2\left(q-i\right)+1} \int_0^{\theta^*} \sin^{2p+1}\theta d\theta.$$

Substituting (2.9), from Lemma 2.3, into (2.14) and integrating, results in (2.12) hence part (i) of the lemma is proved.

(ii) The proof of part (ii) of the lemma follows the same footsteps as part (i).

□

The following lemma is given and will be useful in the simplification of the left hand side of (2.17) and (2.18).

Lemma 2.5. *For $r = 0, 1, 2, \ldots$ and $q = 1, 2, 3, \ldots$ then*

$$\frac{\left(q+\frac{1}{2}\right)_r}{r!} = \frac{1}{4^r}\binom{2r}{r}\prod_{\rho=0}^{q-1}\frac{2r+2\rho+1}{2\rho+1}, \tag{2.15}$$

and for $q = 0$,

$$\frac{\left(\frac{1}{2}\right)_r}{r!} = \frac{1}{4^r}\binom{2r}{r}. \tag{2.16}$$

Proof. For $q = 0$, then

$$\frac{\left(\frac{1}{2}\right)_r}{r!} = \frac{\Gamma\left(r+\frac{1}{2}\right)}{r!\Gamma\left(\frac{1}{2}\right)} = \frac{1}{4^r}\binom{2r}{r}.$$

This result is well known and is also given by Wilf [18].

For $q \geq 1$, let

$$P\left(q\right) := \frac{\left(q+\frac{1}{2}\right)_r}{r!} = \frac{1}{4^r}\binom{2r}{r}\prod_{\rho=0}^{q-1}\frac{2r+2\rho+1}{2\rho+1}$$

then

$$P\left(1\right) = \frac{\left(\frac{3}{2}\right)_r}{r!} = \frac{\left(2r+1\right)}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\frac{\left(\frac{1}{2}\right)_r}{r!},$$

and from (2.16)

$$P\left(1\right) = \frac{2r+1}{4^r}\binom{2r}{r},$$

which satisfies the right hand of (2.15) for $q = 1$.

Consider

$$P\left(q+1\right) = \frac{\left(q+\frac{3}{2}\right)_r}{r!} = \frac{\Gamma\left(q+r+\frac{3}{2}\right)}{r!\Gamma\left(q+\frac{3}{2}\right)}$$
$$= \left(\frac{2q+2r+1}{2q+1}\right)\frac{\Gamma\left(q+r+\frac{1}{2}\right)}{r!\Gamma\left(q+\frac{1}{2}\right)}$$
$$= \left(\frac{2q+2r+1}{2q+1}\right)\frac{\left(q+\frac{1}{2}\right)_r}{r!}$$

and from (2.15)

$$P\left(q+1\right)=\left(\frac{2q+2r+1}{2q+1}\right)\frac{1}{4^{r}}\binom{2r}{r}\prod_{\rho=0}^{q-1}\frac{2r+2\rho+1}{2\rho+1}=\frac{1}{4^{r}}\binom{2r}{r}\prod_{\rho=0}^{q}\frac{2r+2\rho+1}{2\rho+1}$$

hence the lemma is proved. □

For $k=2$ the following theorem now applies.

Theorem 2.6.

(i) *For* $m=2p+1,\, 2\alpha=2q+1;\, p=0,1,2,\dots,\, q=0,1,2,\dots,$

$$\begin{aligned}(2.17)\quad &\sum_{r=0}^{\infty}\frac{\left(q+\frac{1}{2}\right)_{r}}{r!\left(2r+2p+2\right)a^{2r+2p+2}}=\sum_{r=0}^{\infty}\binom{2r}{r}\frac{\prod_{\rho=0}^{q-1}\frac{2r+2\rho+1}{2\rho+1}}{4^{r}\left(2r+2p+2\right)a^{2r+2p+2}}\\ &=T_{0}^{*}\,{}_{2}F_{1}\left[\left.\begin{matrix}p+1, & \frac{1}{2}\left(2q+1\right)\\ & p+2\end{matrix}\right|\frac{1}{a^{2}}\right]\\ &=\frac{1}{2}\left[B\left(\frac{1-2q}{2},p+1\right)-B\left(1-a^{-2};\frac{1-2q}{2},p+1\right)\right]\\ &=\frac{\left(\frac{1}{a}\right)^{2p+2}}{\left(2q-1\right)\cos^{2q-1}\theta^{*}}+\sum_{j=1}^{q-1}\frac{\left(-1\right)^{j}\left(\frac{1}{a}\right)^{2p+2}}{\cos^{2(q-j)-1}\theta^{*}}\prod_{i=1}^{j}\frac{2\left(p-q+i\right)+1}{2\left(q-i\right)+1}\\ &\qquad+\left(-1\right)^{q}\prod_{i=1}^{q}\frac{2\left(p-q+i\right)+1}{2\left(q-i\right)+1}\\ &\qquad\quad\times\left[\frac{\left(-1\right)^{p}}{2^{2p}}\sum_{s=0}^{p}\frac{\left(-1\right)^{s}}{2p-2s+1}\binom{2p+1}{s}\left\{1-\cos\left(\left(2p-2s+1\right)\theta^{*}\right)\right\}\right]\end{aligned}$$

where

$$T_{0}^{*}=\frac{1}{\left(2p+2\right)a^{2p+2}}$$

and $\theta^{*}=\arcsin\left(\frac{1}{a}\right)$.

(ii) *For* $m=2p+2,\, 2\alpha=2q+1;\, p=0,1,2,\dots,\, q=0,1,2,\dots,$

$$\begin{aligned}(2.18)\quad &\sum_{r=0}^{\infty}\frac{\left(q+\frac{1}{2}\right)_{r}}{r!\left(2r+2p+3\right)a^{2r+2p+3}}\\ &\qquad=\sum_{r=0}^{\infty}\binom{2r}{r}\frac{\prod_{\rho=0}^{q-1}\frac{2r+2\rho+1}{2\rho+1}}{4^{r}\left(2r+2p+3\right)a^{2r+2p+3}}\\ &\qquad=T_{0}^{\nabla}\,{}_{2}F_{1}\left[\left.\begin{matrix}\frac{1}{2}\left(2p+3\right), & \frac{1}{2}\left(2q+1\right)\\ \frac{1}{2}\left(2p+5\right) & \end{matrix}\right|\frac{1}{a^{2}}\right]\\ &\qquad=\frac{1}{2}\left[B\left(\frac{1-2q}{2},p+\frac{3}{2}\right)-B\left(1-a^{-2};\frac{1-2q}{2},p+\frac{3}{2}\right)\right]\end{aligned}$$

$$= \frac{\left(\frac{1}{a}\right)^{2p+3}}{(2q-1)\cos^{2q-1}\theta^*} + \sum_{j=1}^{q-1} \frac{(-1)^j \left(\frac{1}{a}\right)^{2p+3}}{\cos^{2(q-j)-1}\theta^*} \prod_{i=1}^{j} \frac{2(p-q+i+1)}{2(q-i)+1}$$
$$+ (-1)^q \prod_{i=1}^{q} \frac{2(p-q+i+1)}{2(q-i)+1} \left\{ \frac{1}{2^{2p+1}} \left[\binom{2p+1}{p} \theta^* \right.\right.$$
$$\left.\left. - (-1)^p \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+2} \binom{2p+2}{s} \sin\left((2p-2s+2)\theta^*\right) \right] \right\},$$

where

$$T_0^{\nabla} = \frac{1}{(2p+3)\, a^{2p+3}}$$

and $\theta^* = \arcsin\left(\frac{1}{a}\right)$.

The proof of Theorem 2.6 follows directly from Lemma 2.4 , (2.14) and Lemma 2.5.

Some examples will now be given expressing π and other constants in terms of an infinite series.

3. Illustrative Examples

Example 3.1. From (2.17) with $q = 2$, $a = 2$ and $\theta^* = \frac{\pi}{6}$, we have

$$\frac{2}{\sqrt{3}}\left(\frac{8}{3} - 2(2p-1)\right) + 8(-1)^p (2p-1)(2p+1)$$
$$\times \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+1} \binom{2p+1}{s} \left\{1 - \cos(2p-2s+1)\frac{\pi}{6}\right\}$$
$$= \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(2r+1)(2r+3)}{16^r (r+p+1)}.$$

Hence, for $p = 7$,

$$\sqrt{3} = \frac{3 \cdot 2^{22}}{7 \cdot 163 \cdot 6367} - \frac{3^2 \cdot 11}{2^6 \cdot 7 \cdot 163 \cdot 6376} \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(2r+1)(2r+3)}{16^r (r+8)}.$$

Example 3.2. From (2.18) with $q = 2$, $a = \frac{2}{\sqrt{3}}$ and $\theta^* = \frac{\pi}{3}$, we have

$$8 - 4p + \frac{16p(p+1)}{\left(\sqrt{3}\right)^{2p+3}} \left[\binom{2p+1}{p} \frac{\pi}{3} \right.$$
$$\left. - (-1)^p \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+2} \binom{2p+2}{s} \sin(2p-2s+2)\frac{\pi}{3} \right]$$
$$= \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(2r+1)(2r+3)}{(2r+2p+3)} \left(\frac{3}{16}\right)^r$$

or rearranging,

$$\frac{\pi}{3}=\frac{3^{p+\frac{3}{2}}}{16p(p+1)\binom{2p+1}{p}}\left\{4p-8+\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)}{(2r+2p+3)}\left(\frac{3}{16}\right)^{r}\right\}$$
$$+\frac{(-1)^{p}}{\binom{2p+1}{p}}\sum_{s=0}^{p}\frac{(-1)^{s}}{2p-2s+2}\binom{2p+2}{s}\sin(2p-2s+2)\frac{\pi}{3},$$

and for $p=2$

$$\frac{20\sqrt{3}\pi}{81}=1+\frac{1}{16}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)}{2r+7}\left(\frac{3}{16}\right)^{r}.$$

For $p=2$, $a=\sqrt{2}$

$$\pi=\frac{8}{3}+\frac{1}{3\sqrt{2}}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)}{2r+5}\left(\frac{1}{8}\right)^{r}.$$

Example 3.3. For $q=3$, $p=2$ and $a=\sqrt{2}$,

$$\pi=\frac{52}{15}-\frac{1}{30\sqrt{2}}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)(2r+5)}{2r+7}\left(\frac{1}{8}\right)^{r}.$$

Example 3.4. For $q=4$, $p=3$ and $a=2$,

$$\pi=\frac{1712\sqrt{3}}{945}+\frac{1}{8960}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)(2r+5)(2r+7)}{(2r+9)16^{r}}.$$

Example 3.5. For $q=5$, $p=0$ and $a=\sqrt{5}$,

$$\sqrt{5}=\frac{2^{11}}{3\cdot5\cdot193\cdot2731}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+5)(2r+7)(2r+9)(2r+11)}{20^{r}}.$$

Example 3.6. For $q=6$, $p=5$ and $a=\sqrt{2}$,

$$\pi=\frac{2^{3}\cdot1289}{5\cdot7\cdot9\cdot11}+\frac{1}{5\cdot16\cdot829\cdot\sqrt{2}}\sum_{r=0}^{\infty}\left[\binom{2r}{r}\right.$$
$$\left.\times\frac{(2r+1)(2r+3)(2r+5)(2r+7)(2r+9)(2r+11)}{(2r+13)8^{r}}\right].$$

Example 3.7. For $q=4$ $\left(\alpha=\frac{9}{2}\right)$, $p=59$ $(m=120)$ and $a=2$,

$$\pi=\frac{\Omega_1}{\Omega_2\sqrt{3}}+\frac{1}{\Omega_3}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)(2r+5)(2r+7)}{(2r+121)16^{r}},$$

where

$$\begin{aligned}\Omega_1&=15604102274295581508678435968572864501995513795052733\\&=(46042305118509401202197)(338907929004245243145594887689),\end{aligned}$$

$$\Omega_2=2^{6}\cdot3^{5}\cdot5^{2}\cdot7^{2}\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37\cdot41\cdot43\cdot47\cdot53\cdot59$$
$$\cdot61\cdot67\cdot71\cdot73\cdot79\cdot83\cdot89\cdot97\cdot101\cdot103\cdot107\cdot109\cdot113$$

and

$$\Omega_3 = 2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 59 \\ \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113.$$

In this case the first term of the sum gives π accurate to over forty decimal places.

Other particular values of constants may be obtained from (2.13).

Example 3.8. For $a = 2$, $m = 10$, $k = 1$ and $\alpha = 7$

$$\ln 2 = \frac{27947}{2^7 \cdot 3^2 \cdot 5 \cdot 7} + \frac{1}{2^{12} \cdot 3 \cdot 5 \cdot 7} \sum_{r=0}^{\infty} \binom{r+6}{r} \frac{1}{(r+11)\, 2^r}.$$

It is of some interest to note that from (2.18) for $m = 0$ and $\alpha = w > 1$, integer, we may eventually write, after integration by parts, and using (2.2)

$$\ln\left(\frac{a+1}{a-1}\right) = \frac{2^w (w-1)!}{(2w-3)!!} \left[\sum_{r=0}^{\infty} \binom{r+w-1}{r} \frac{1}{(2r+1)\, a^{2r+1}} \right. \\ \left. - \frac{1}{a\ (2w-1)} \sum_{j=1}^{w-1} \frac{1}{2^j} \left(\frac{a^2}{a^2-1}\right)^{w-j} \prod_{\nu=1}^{j} \frac{2w-2\nu+1}{w-\nu} \right],$$

where $(2w-3)!! = (2w-3)(2w-5)\cdots 5 \cdot 3 \cdot 1$, and $a > 1$.
For $a = 11$ and $w = 7$, we have

$$\ln\left(\frac{6}{5}\right) = \frac{2^{11}}{3 \cdot 7 \cdot 11^2} \sum_{r=0}^{\infty} \binom{r+6}{r} \frac{1}{(2r+1)\, 121^r} - \frac{11 \cdot 179 \cdot 17047711}{2^9 \cdot 3^8 \cdot 5^6}.$$

Remark 3.1. In the degenerative case of $w = 1$ then we obtain the well-known formula as listed in Abramowitz and Stegun [1], namely

$$\ln\left(\frac{a+1}{a-1}\right) = 2 \sum_{r=0}^{\infty} \frac{1}{(2r+1)\, a^{2r+1}}.$$

4. SOME ESTIMATES

It is useful to be able to obtain some estimates of the representation of the series (2.2). This is done in the following theorems.

Theorem 4.1. *Given that*

$$S(a,k,\alpha,m) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!\,(rk+m+1)\, a^{rk+m+1}}, \tag{4.1}$$

then

$$\begin{aligned} &\frac{1}{(m+1)\, a^{m+1}} \\ &< S(a,k,\alpha,m) \\ &\le \begin{cases} \frac{1}{((mq+1)a^{mq+1})^{\frac{1}{q}} k^{\frac{1}{p}}} \left[B\left(1-\alpha p, \frac{1}{k}\right) - B\left(1-a^{-k}; 1-\alpha p, \frac{1}{k}\right)\right]^{\frac{1}{p}}, \\ \left(\frac{a^k}{a^k-1}\right)^{\alpha} \frac{1}{(m+1)a^{m+1}}, \qquad a > 1, \end{cases} \end{aligned} \tag{4.2}$$

for real numbers p *and* q *where* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $B(s,t)$ *is the classical Beta function and* $B(z;s,t)$ *is the incomplete Beta function as described in Theorem 2.1.*

Proof. Let $f(x) = \frac{1}{(1-x^k)^\alpha}$ and $g(x) = x^m$. Since $|f(x)|^p$ and $|g(x)|^q$ are integrable functions defined on $x \in \left[0, \frac{1}{a}\right]$, then by Hölder's integral inequality [14]

$$S(a,k,\alpha,m) \le \left(\int_0^{\frac{1}{a}} x^{mq} dx\right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{a}} \frac{dx}{(1-x^k)^{\alpha p}}\right)^{\frac{1}{p}}.$$

Now

$$\int_0^{\frac{1}{a}} \frac{dx}{(1-x^k)^{\alpha p}} = \frac{1}{k}\left[B\left(1-\alpha p, \frac{1}{k}\right) - B\left(1-a^{-k}; 1-\alpha p, \frac{1}{k}\right)\right]$$

by the substitution $u = 1 - x^k$, and hence

$$S(a,k,\alpha,m) \le \frac{1}{\left((mq+1)\, a^{mq+1}\right)^{\frac{1}{q}} k^{\frac{1}{p}}} \left[B\left(1-\alpha p, \frac{1}{k}\right) - B\left(1-a^{-k}; 1-\alpha p, \frac{1}{k}\right)\right]^{\frac{1}{p}}.$$

The lower bound on $S(a,k,\alpha,m)$ is $\frac{1}{(m+1)a^{m+1}}$ since the sum (2.2) is one of positive terms.

The second part of the inequality (4.2) is obtained from

$$\int_{x_0}^{x_1} |f(x)\, g(x)|\, dx \le ess \sup_{x\in[x_0,x_1]} |f(x)| \int_{x_0}^{x_1} |g(x)|\, dx.$$

Since $f(x)$ is monotonic on $x \in \left[0, \frac{1}{a}\right]$,

$$ess \sup_{x\in\left[0,\frac{1}{a}\right]} \left\{\frac{1}{(1-x^k)^\alpha}\right\} = \left(\frac{a^k}{a^k-1}\right)^\alpha$$

and

$$\int_0^{\frac{1}{a}} x^m dx = \frac{1}{(m+1)\, a^{m+1}},$$

hence

$$\frac{1}{(m+1)\, a^{m+1}} < S(a,k,\alpha,m) \le \left(\frac{a^k}{a^k-1}\right)^\alpha \frac{1}{(m+1)\, a^{m+1}}.$$

The result (4.2) follows and the theorem is proved. □

The next theorem develops an inequality of (4.1) based on the pre-Grüss result.

Theorem 4.2. *For* $a > 1$,

$$\left|S(a,k,\alpha,m) - \frac{1}{k(m+1)\, a^m}\left\{B\left(1-\alpha, \frac{1}{k}\right) - B\left(1-a^{-k}; 1-\alpha, \frac{1}{k}\right)\right\}\right| \le \frac{m}{2(m+1)\, a^{m+1}\sqrt{2m+1}}\left[\left(\frac{a^k}{a^k-1}\right)^\alpha - 1\right]. \tag{4.3}$$

Proof. Define

$$T(g,f) := \frac{1}{x_1-x_0}\int_{x_0}^{x_1} f(x)\, g(x)\, dx - \frac{1}{x_1-x_0}\int_{x_0}^{x_1} f(x)\, dx \cdot \frac{1}{x_1-x_0}\int_{x_0}^{x_1} g(x)\, dx$$

for $f(x)$ and $g(x)$ integrable functions, as given in Theorem 4.1, and defined on $x \in \left[0, \frac{1}{a}\right]$, then the pre-Grüss inequality [13] states that

$$|T(g,f)| \le \frac{\Gamma - \gamma}{2}\left[T(g,g)\right]^{\frac{1}{2}}$$

for $\gamma \leq f(x) \leq \Gamma$.

Now, for $x \in \left[0, \frac{1}{a}\right]$

$$\gamma = 1 \leq f(x) = \frac{1}{(1-x^k)^\alpha} \leq \left(\frac{a^k}{a^k-1}\right)^\alpha = \Gamma,$$

$$\begin{aligned} T(g,f) &= a\int_0^{\frac{1}{a}} \frac{x^m}{(1-x^k)^\alpha}dx - a^2\int_0^{\frac{1}{a}} \frac{dx}{(1-x^k)^\alpha}\int_0^{\frac{1}{a}} x^m dx \\ &= aS(a,k,\alpha,m) - \frac{a^2}{(m+1)a^{m+1}k}\left[B\left(1-\alpha,\frac{1}{k}\right) - B\left(1-a^{-k};1-\alpha,\frac{1}{k}\right)\right]. \end{aligned}$$

In a similar fashion

$$[T(g,g)]^{\frac{1}{2}} = \frac{m}{(m+1)a^m\sqrt{2m+1}}.$$

Combining $T(g,f)$ and $[T(g,g)]^{\frac{1}{2}}$ gives us the result (4.3) after a little algebraic simplification. $\square$

Open Problem 1. From Example 3.2, we have that

$$\begin{aligned} U_\infty = \frac{3^{p+\frac{3}{2}}}{16p(p+1)\binom{2p+1}{p}}&\left\{4p-8+\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(2r+1)(2r+3)}{(2r+2p+3)}\left(\frac{3}{16}\right)^r\right\} \\ &+\frac{(-1)^p}{\binom{2p+1}{p}}\sum_{s=0}^{p}\frac{(-1)^s}{2p-2s+2}\binom{2p+2}{s}\sin(2p-2s+2)\frac{\pi}{3} = \frac{\pi}{3}. \end{aligned}$$

Now let us consider the following. For a finite positive integer R let

$$\begin{aligned} U_R = \frac{3^{p+\frac{3}{2}}}{16p(p+1)\binom{2p+1}{p}}&\left\{4p-8+\sum_{r=0}^{R}\binom{2r}{r}\frac{(2r+1)(2r+3)}{(2r+2p+3)}\left(\frac{3}{16}\right)^r\right\} \\ &+\frac{(-1)^p}{\binom{2p+1}{p}}\sum_{s=0}^{p}\frac{(-1)^s}{2p-2s+2}\binom{2p+2}{s}\sin(2p-2s+2)\frac{\pi}{3}, \\ U_R &= V+W, \end{aligned}$$

in fact

$$U_R < \frac{\pi}{3}.$$

For a fixed positive integer R, it can be shown, by standard analysis methods, that

$$\begin{aligned} \lim_{p->\infty} V = \lim_{p->\infty}&\frac{3^{p+\frac{3}{2}}}{16p(p+1)\binom{2p+1}{p}} \\ &\times\left\{4p-8+\sum_{r=0}^{R}\binom{2r}{r}\frac{(2r+1)(2r+3)}{(2r+2p+3)}\left(\frac{3}{16}\right)^r\right\} = 0 \end{aligned}$$

and as $p->\infty$ uniformly

$$W = \frac{(-1)^p}{\binom{2p+1}{p}}\sum_{s=0}^{p}\frac{(-1)^s}{2p-2s+2}\binom{2p+2}{s}\sin(2p-2s+2)\frac{\pi}{3} \approxeq \frac{\pi}{3}.$$

This implies that for $p- > \infty,\ W \approxeq U_\infty$ and

$$W = \frac{(-1)^p}{\binom{2p+1}{p}} \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+2} \binom{2p+2}{s} \sin(2p-2s+2)\frac{\pi}{3} \approxeq \frac{\pi}{3}.$$

An open problem is to prove, or provide a contradiction to,

$$\lim_{p->\infty} W = \lim_{p->\infty} \left\{ \frac{(-1)^p}{\binom{2p+1}{p}} \sum_{s=0}^{p} \frac{(-1)^s}{2p-2s+2} \binom{2p+2}{s} \sin(2p-2s+2)\frac{\pi}{3} \right\} = \frac{\pi}{3}.$$

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