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ON RELATIONS OF COEFFICIENT CONDITIONS

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ABSTRACT. We analyze the relations of three coefficient conditions of different type implying one by one the absolute convergence of the Haar series. Furthermore we give a sharp condition which guaranties the equivalence of these coefficient conditions.

Key words and phrases: Haar series, Absolute convergence, Equivalence of coefficient conditions.

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1. INTRODUCTION

A known result of P.L. Ul'janov [4] asserts that the condition

$$\sigma_1 := \sum_{n=3}^{\infty} \frac{a_n}{\sqrt{n}} < \infty \quad (a_n \geq 0) \tag{1.1}$$

implies the absolute convergence of the Haar series, i.e.

$$\sum_{m=0}^{\infty} \sum_{k=1}^{2^m} \left| b_m^{(k)} \chi_m^{(k)}(x) \right| \equiv \sum_{n=0}^{\infty} |a_n \, \chi_n(x)| < \infty$$

almost everywhere in $(0,1)$. He also verified, among others, that if the sequence $\{a_n\}$ is monotone then the condition (1.1) is not only sufficient, but also necessary to the absolute convergence of the Haar series.

In [1] we verified that if the condition

$$\sigma_2 := \sum_{m=1}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{\frac{1}{2}} < \infty \tag{1.2}$$

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holds then the Haar series is absolute (C, α)-summable for any $\alpha \geq 0$, consequently the condition (1.2) also guarantees the absolute convergence of the Haar series.

Recently, in [3], we showed that if the sequence $\{a_n\}$ is only *locally quasi decreasing*, i.e. if

$$a_n \leq K\, a_m \quad \text{for} \quad m \leq n \leq 2m \quad \text{and for all} \quad m,$$

and the Haar series is absolute $(C,\, \alpha \geq 0)$-summable almost everywhere, then (1.2) holds.

Here and in the sequel, K and K_i will denote positive constants, not necessarily the same at each occurrence. Furthermore we shall say that a sequence $\{a_n\}$ is *quasi decreasing* if

$$(0 \leq) a_n \leq K\, a_m$$

holds for any $n \geq m$. This will be denoted by $\{a_n\} \in QDS$, and if the sequence $\{a_n\}$ is a locally quasi decreasing, then we use the short notion $\{a_n\} \in LQDS$.

P.L. Ul'janov [5], implicitly, gave a further condition in the form

$$\sigma_3 := \sum_{m=3}^{\infty} \frac{1}{m(\log m)^{\frac{1}{2}}} \left\{ \sum_{n=m}^{\infty} a_n^2 \right\}^{\frac{1}{2}} < \infty \tag{1.3}$$

which also implies the absolute convergence of the Haar series.

These results propose the question: What is the relation among these conditions?

We shall show that the condition (1.3) claims more than (1.2), and (1.2) demands more than (1.1); and in general, they cannot be reversed. In order to get an opposite implication, a certain monotonicity condition on the sequence $\{a_n\}$ is required.

2. Results

We establish the following theorem.

Theorem 2.1. *Suppose that $a := \{a_n\}$ is a sequence of nonnegative numbers. Then the following assertions hold:*

$$\sigma_1 \leq K\, \sigma_2, \tag{2.1}$$

and if $a \in LQDS$ then

$$\sigma_2 \leq K\, \sigma_1. \tag{2.2}$$

Similarly

$$\sigma_2 \leq K\, \sigma_3, \tag{2.3}$$

and if the sequence $\{A_m\}$ defined by

$$A_m := \left\{ \sum_{k=2^m+1}^{2^{m+1}} a_k^2 \right\}^{\frac{1}{2}}$$

belongs to QDS then

$$\sigma_3 \leq K\, \sigma_2. \tag{2.4}$$

Finally

$$\sigma_1 \leq K\, \sigma_3, \tag{2.5}$$

and if the sequence $\{n\, a_n^2\} \in QDS$ then

$$\sigma_3 \leq K\, \sigma_1. \tag{2.6}$$

Corollary 2.2. *If the sequence $\{n\, a_n^2\} \in QDS$ then the conditions (1.1), (1.2) and (1.3) are equivalent.*

Next we show that the assumption $\{n\, a_n^2\} \in QDS$ in a certain sense is sharp. Namely if we claim only that the sequence $\{n^\alpha\, a_n^2\} \in QDS$ with $\alpha < 1,$ then already the implication (1.1) ⇒ (1.3), in general, does not hold.

Proposition 2.3. *If* $(0\ \leq)\, \alpha\ <\ 1$ *then there exists a sequence* $\{a_n\}$ *such that the sequence* $\{n^\alpha\, a_n^2\} \in QDS$*, furthermore*

$$\sigma_1 < \infty \quad \textit{but} \quad \sigma_3 = \infty.$$

Finally we verify the following.

Proposition 2.4. *The requirements*

$$\{n\, a_n^2\} \in QDS \tag{2.7}$$

and the following two assumptions jointly

$$\{A_m\} \in QDS \quad \textit{and} \quad \{a_n\} \in LQDS \tag{2.8}$$

are equivalent.

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3. LEMMA

We require the following lemma being a special case of a theorem proved in [2, Satz] appended with the inequality (3.2) which was also verified, in the same paper, in the proof of the "Hilfssatz" (see p. 217).

Lemma 3.1. *The inequality (1.3) holds if and only if there exists a nondecreasing sequence* $\{\mu_n\}$ *of positive numbers with the properties*

$$\sum_{n=1}^{\infty} \frac{1}{n\, \mu_n} < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n^2\, \mu_n < \infty. \tag{3.1}$$

Furthermore

$$\sum_{n=3}^{\infty} \frac{1}{n (\log n)^{\frac{1}{2}}} \left\{ \sum_{k=n}^{\infty} a_k^2 \right\}^{\frac{1}{2}} \leq K \left\{ \sum_{n=3}^{\infty} a_n^2\, \mu_n \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n\, \mu_n} \right\}^{\frac{1}{2}} \tag{3.2}$$

also holds.

4. PROOFS

Proof of Theorem 2.1. The inequality (2.1) can be verified by then Hölder inequality. Namely

$$\sigma_1 = \sum_{m=1}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{a_n}{\sqrt{n}} \leq \sum_{m=1}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n} \right\}^{\frac{1}{2}} \leq \sigma_2.$$

To prove the inequality (2.2) we utilize the monotonicity assumption and thus we get that

$$\sigma_2 \leq K \sum_{m=1}^{\infty} 2^{m/2} a_{2^m+1} \leq K_1 \sum_{m=1}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{\sqrt{n}} a_n = K_1 \sigma_1.$$

The inequality (2.3) also comes via the Hölder inequality. Let $R_m := \left\{\sum_{n=m}^{\infty} a_n^2\right\}^{\frac{1}{2}}$. Then

$$\begin{aligned}
\sigma_2 &= \sum_{\nu=0}^{\infty}\sum_{m=2^\nu}^{2^{\nu+1}-1}\left\{\sum_{n=2^m+1}^{2^{m+1}} a_n^2\right\}^{\frac{1}{2}} \\
&\le \sum_{\nu=0}^{\infty} 2^{\nu/2}\left\{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_n^2\right\}^{\frac{1}{2}} \\
&\le \sum_{\nu=0}^{\infty} 2^{\nu/2}\left\{\sum_{n=2^{2^\nu}+1}^{\infty} a_n^2\right\}^{\frac{1}{2}} \\
&\le R_3 + K\sum_{\nu=1}^{\infty}\sum_{n=2^{2^{\nu-1}}+1}^{2^{2^\nu}} \frac{1}{n(\log n)^{\frac{1}{2}}} R_{2^{2^\nu}+1} \\
&\le K_1 \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\frac{1}{2}}} R_n = K_1\sigma_3.
\end{aligned}$$

In order to prove (2.4) first we define a nondecreasing sequence $\{\mu_n\}$ as follows. Let

$$\mu_n := \max_{1\le k\le m} A_k^{-1} \quad \text{for} \quad 2^m < n \le 2^{m+1}, \qquad m = 1,2,\dots,$$

furthermore let $\mu_1 = \mu_2 = \mu_3$. It is clear by $\{A_m\} \in QDS$ that

$$A_m^{-1} \le \mu_{2^{m+1}} \le K\, A_m^{-1} \quad (m \ge 1), \tag{4.1}$$

holds. Hence we obtain by (1.2) and (4.1) that

$$\sum_{m=1}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} a_n^2\,\mu_n \le K\,\sigma_2 < \infty \tag{4.2}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n\,\mu_n} &\le K\sum_{n=3}^{\infty}\frac{1}{n\,\mu_n} \\
&= K\sum_{m=1}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\frac{1}{n\,\mu_n} \\
&\le K_1\sum_{m=1}^{\infty}\frac{1}{\mu_{2^{m+1}}} \\
&\le K_1\sum_{m=1}^{\infty} A_m = K_1\sigma_2 < \infty.
\end{aligned} \tag{4.3}$$

Finally, using the inequality (3.2), the estimations (4.2) and (4.3) clearly imply the statement (2.4).

The assertion (2.5) is an immediate consequence of (2.1) and (2.3).

The proof of the declaration (2.6) is analogous to that of (2.4). The assumption $\{n\,a_n^2\} \in QDS$ enables us to define again a nondecreasing sequence $\{\mu_n\}$ satisfying the inequalities in

(3.1). We can clearly assume that all $a_k > 0$, otherwise (2.6) is trivial if $\{n\, a_n^2\} \in QDS$. Let for $n \geq 3$

$$\mu_n := \max_{1\leq k\leq n} \frac{1}{a_k\sqrt{k}}, \quad \text{and} \quad \mu_1 = \mu_2 = \mu_3.$$

The definition of μ_n and the assumption $\{n\, a_n^2\} \in QDS$ certainly imply that

$$\frac{1}{a_n\sqrt{n}} \leq \mu_n \leq \frac{K}{a_n\sqrt{n}} \tag{4.4}$$

is valid. The definition of σ_1 given in (1.1) and (4.4) convey the estimations

$$\sum_{n=3}^{\infty} a_n^2\, \mu_n \leq K \sum_{n=3}^{\infty} \frac{a_n}{\sqrt{n}} \leq K\, \sigma_1 < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n\, \mu_n} \leq K \sum_{n=3}^{\infty} \frac{1}{n\, \mu_n} = K \sum_{n=3}^{\infty} \frac{a_n}{\sqrt{n}} = K\, \sigma_1 < \infty.$$

These estimations and (3.2) verify (2.6).

Herewith the whole theorem is proved. □

Proof of Corollary 2.2. The inequalities (2.1), (2.3) and (2.6) proved in the theorem obviously deliver the assertion of the corollary. The proof is ready. □

Proof of Proposition 2.3. Setting

$$\nu_m := 2^{2^m}, \quad \varepsilon_m := 2^{-m/2} \nu_{m+1}^{\frac{\alpha-1}{2}}$$

and

$$a_n^2 := \varepsilon_m^2\, n^{-\alpha} \quad \text{if} \quad \nu_m < n \leq \nu_{m+1}, \quad m = 0, 1, \dots$$

Then

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{a_n}{\sqrt{n}} &= \sum_{m=0}^{\infty} \varepsilon_m \sum_{n=\nu_m+1}^{\nu_{m+1}} n^{-\frac{1+\alpha}{2}} \\ &\leq \sum_{m=0}^{\infty} \varepsilon_m\, \nu_{m+1}^{\frac{1-\alpha}{2}} = \sum_{m=0}^{\infty} 2^{-m/2} < \infty,\end{aligned}$$

however, with $R_n := \left\{\sum\limits_{k=n}^{\infty} a_k^2\right\}^{\frac{1}{2}}$,

$$\begin{aligned}\sigma_3 &= \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\frac{1}{2}}} R_n \\ &= \sum_{m=0}^{\infty} \sum_{n=\nu_m+1}^{\nu_{m+1}} \frac{1}{n(\log n)^{\frac{1}{2}}} R_n \\ &\geq \frac{1}{4} \sum_{m=0}^{\infty} R_{\nu_{m+1}} 2^{m/2},\end{aligned}$$

furthermore

$$R_{\nu_m}^2 \geq \sum_{k=m}^{\infty} \sum_{n=\nu_k+1}^{\nu_{k+1}} a_n^2 = \sum_{k=m}^{\infty} \varepsilon_k^2 \sum_{n=\nu_k+1}^{\nu_{k+1}} k^{-\alpha}$$

$$\geq \frac{1}{K} \sum_{k=m}^{\infty} \varepsilon_k^2 \, \nu_{k+1}^{1-\alpha} = \frac{1}{K} \sum_{k=m}^{\infty} 2^{-k} \geq \frac{1}{K} 2^{-m}.$$

From the last two estimations we clearly get that $\sigma_3 = \infty$, as stated.

The proof is complete. □

Proof of Proposition 2.4. First we prove that the assumption (2.7) implies both properties claimed in (2.8). Namely by $\{n\, a_n^2\} \in QDS$ we get that if $\mu > m$ then

$$A_m^2 = \sum_{n=2^m+1}^{2^{m+1}} \frac{a_n^2 n}{n} \geq \frac{1}{2^{m+1}} 2^m \frac{1}{K} a_{2^{m+1}}^2 2^{m+1}$$

$$\geq \frac{1}{2K^2} a_{2^\mu}^2 2^\mu \geq \frac{1}{2K^3} \sum_{n=2^\mu+1}^{2^{\mu+1}} a_n^2$$

$$= \frac{1}{2K^3} A_\mu^2,$$

i.e. $\{n\, a_n^2\} \in QDS \Rightarrow \{A_n\} \in QDS$ holds.

The implications $\{n\, a_n^2\} \in QDS \Rightarrow \{a_n\} \in QDS \Rightarrow \{a_n\} \in LQDS$ are trivial.

To prove the implication (2.8) ⇒ (2.7) we first prove by $\{a_n\} \in LQDS$ that if $\mu > m$ then

$$\sum_{k=2^m+1}^{2^{m+1}} a_k^2 \leq K\, 2^m\, a_{2^m}^2$$

and

$$\sum_{k=2^{\mu-1}+1}^{2^\mu} a_k^2 \geq 2^{\mu-1} \frac{1}{K} a_{2^\mu}^2,$$

thus by $\{A_n\} \in QDS$ we obtain that

$$2^\mu\, a_{2^\mu}^2 \leq K_1\, 2^m\, a_{2^m}^2$$

holds, whence $\{n\, a_n^2\} \in QDS$ plainly follows.

The proof is ended. □

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