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SOME HARDY TYPE INTEGRAL INEQUALITIES

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ABSTRACT. In this note, we obtain some new generalizations of the Hardy's integral inequality by using a fairly elementary analysis. These inequalities generalize some known results and simplify the proofs of some existing results.

Key words and phrases: Hardy integral inequalities and Hölder's inequality.

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1. INTRODUCTION

In [4], Hardy proved the following inequality. If $p > 1,\ f \geq 0$ and

$$F(x) = \int_0^x f(t)dt,$$

then

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < q^p \int_0^\infty f^p(x)dx \tag{1.1}$$

unless $f \equiv 0$. The constant $q = p\,(p-1)^{-1}$ is the best possible. This inequality plays an important role in analysis and its applications. It is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid

$$\int_a^b \left(\frac{F}{x}\right)^p dx < q^p \int_a^b f^p(x)dx, \tag{1.2}$$

where $0 < \int_0^\infty f^p(x)dx < \infty$. The classical Hardy inequality asserts that if $p > 1$ and f is a nonnegative measurable function on (a, b), then (1.2) is true unless $f \equiv 0$ a.e. in (a, b), where the constant here is best possible. This inequality remains true provided that $0 < a < b < \infty$.

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In particular, Hardy [3] in 1928 gave a generalized form of the inequality (1.1) when he showed that for any $m \neq 1, p > 1$ and any integrable function $f(x) \geq 0$ on $(0, \infty)$ for which

$$F(x) = \begin{cases} \int_0^x f(t)dt & \text{for} \quad m > 1, \\ \int_x^\infty f(t)dt & \text{for} \quad m < 1, \end{cases}$$

then

$$\int_0^\infty x^{-m} F^p(x)dx < \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m} \left[xf(x)\right]^p dx \tag{1.3}$$

unless $f \equiv 0$, where the constant is also best possible.

Because of their fundamental importance in the discipline over the years much effort and time has been devoted to the improvement and generalization of Hardy's inequalities (1.1), (1.2) and (1.3). These include, among others, the works in [1] – [9].

The objective of this paper is to obtain further generalizations of the classical Hardy integral inequality which will be useful in applications by using some elementary methods of analysis. Throughout this paper, the left-hand sides of the inequalities exist if the right-hand sides exist.

2. MAIN RESULTS

The following theorems are the main results of the present paper.

Theorem 2.1. *Let $p > 1,\ m > 1$ be constants. Let $f(x)$ be a nonnegative and integrable function on $(0, \infty)$ and let $z(x)$ be differentiable in $(0, \infty)$ with $z'(x) > 0$ and $z(0_+) > 0$. Let $w(x)$ and $r(x)$ be positive and absolutely continuous functions on $(0, \infty)$. Let*

$$1 - \frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)} + \frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)} \geq \frac{1}{\lambda} > 0.$$

Let $a \in (0, \infty)$ be fixed and set

$$F(x) := \frac{1}{r(x)} \int_a^x \frac{r(t)z'(t)f(t)}{z(t)} dt, \qquad x \in (0, \infty).$$

Then the inequality

$$\int_a^b w(x)\frac{z'(x)}{z^m(x)} F^p(x)\,dx \leq \left(\frac{\lambda p}{m-1}\right)^p \int_a^b w(x)\frac{z'(x)}{z^m(x)} f^p(x)\,dx \tag{2.1}$$

holds for all $b \geq a$.

Proof. Integrating the left-hand side of inequality (2.1) by parts gives

$$\int_a^b w(x)\frac{z'(x)}{z^m(x)} F^p(x)\,dx = w(b)\frac{[z(b)]^{-m+1}}{-m+1} F^p(b) + \frac{1}{m-1}\int_a^b z^{-m+1} w' F^p dx$$
$$- \frac{p}{m-1}\int_a^b z^{-m+1} w \frac{r'(x)}{r(x)} F^p dx + \frac{p}{m-1}\int_a^b w \frac{z'(x)}{z^m(x)} F^{p-1} f dx.$$

Since $m>1$ and $F(b)\geq 0$, we have

$$\begin{aligned}\int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\left[1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\right]dx \\ = w(b)\frac{[z(b)]^{-m+1}}{-m+1}F^p(b)+\frac{p}{m-1}\int_a^b w\frac{z'(x)}{z^m(x)}F^{p-1}fdx \\ \leq \frac{p}{m-1}\int_a^b w\frac{z'(x)}{z^m(x)}F^{p-1}fdx.\end{aligned}$$

Here, using the assumption on λ, we have

$$\begin{aligned}&\int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\frac{1}{\lambda}dx \\ &\quad\leq \int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\left[1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\right]dx \\ &\quad\leq \frac{p}{m-1}\int_a^b w\frac{z'(x)}{z^m(x)}F^{p-1}fdx.\end{aligned}$$

By Hölder's inequality,

$$\int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx\leq\frac{\lambda p}{m-1}\left(\int_a^b w\frac{z'(x)}{z^m(x)}F^pdx\right)^{\frac{1}{q}}\left(\int_a^b w\frac{z'(x)}{z^m(x)}f^pdx\right)^{\frac{1}{p}}$$

and thus on simplification, we have

$$\int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx\leq\left(\frac{\lambda p}{m-1}\right)^p\int_a^b w\frac{z'(x)}{z^m(x)}f^pdx.$$

This proves the theorem. □

Theorem 2.2. *Let p,m,f,z,z',w and r be as in Theorem 2.1. Let*

$$1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}-\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\geq\frac{1}{\lambda}>0.$$

Let $a\in(0,\infty)$ be fixed and set

$$F(x):=r(x)\int_a^x\frac{z'(t)f(t)}{z(t)r(t)}dt,\qquad x\in(0,\infty).$$

Then the inequality

$$\int_a^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx\leq\left(\frac{\lambda p}{m-1}\right)^p\int_a^b w(x)\frac{z'(x)}{z^m(x)}f^p(x)\,dx \tag{2.2}$$

holds for all $b\geq a$.

Proof. This is similar to the proof of the Theorem 2.1. □

Theorem 2.3. *Let p,m,f,z,z',w and r be as in Theorem 2.1. Let*

$$1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\geq\frac{1}{\lambda}>0.$$

Let $a\in(0,\infty)$ be fixed and set

$$F(x):=\frac{1}{r(x)}\int_{\frac{x}{2}}^x\frac{r(t)z'(t)f(t)}{z(t)}dt,\qquad x\in(0,\infty).$$

Then the inequality

$$\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx \le \left(\frac{\lambda p}{m-1}\right)^p \int_0^b w(x)\frac{[z(x)]^{p-m}}{[z'(x)]^{\frac{p}{q}}}|g(x)|^p\,dx \tag{2.3}$$

holds where

$$g(x)=\frac{1}{r(x)}\left(\frac{r(x)z'(x)f(x)}{z(x)}-\frac{1}{2}\frac{r(\frac{x}{2})z'(\frac{x}{2})f(\frac{x}{2})}{z(\frac{x}{2})}\right).$$

Proof. Upon integrating by parts we have

$$\begin{aligned}\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx &= w(b)\frac{[z(b)]^{-m+1}}{-m+1}F^p(b)+\frac{1}{m-1}\int_0^b z^{-m+1}w'F^p dx\\ &-\frac{p}{m-1}\int_0^b z^{-m+1}w\frac{r'(x)}{r(x)}F^p dx+\frac{p}{m-1}\int_0^b z^{-m+1}wF^{p-1}|g(x)|\,dx,\end{aligned}$$

where

$$g(x)=\frac{1}{r(x)}\left(\frac{r(x)z'(x)f(x)}{z(x)}-\frac{1}{2}\frac{r(\frac{x}{2})z'(\frac{x}{2})f(\frac{x}{2})}{z(\frac{x}{2})}\right).$$

Since $m>1$ and $F(b)\ge 0$ we have

$$\begin{aligned}\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)&\left[1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\right]dx\\ &= w(b)\frac{[z(b)]^{-m+1}}{-m+1}F^p(b)+\frac{p}{m-1}\int_0^b w\frac{z'(x)}{z^m(x)}F^{p-1}f dx\\ &\le \frac{p}{m-1}\int_0^b z^{-m+1}wF^{p-1}|g(x)|\,dx.\end{aligned}$$

By the assumption on λ, we have

$$\begin{aligned}&\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\frac{1}{\lambda}dx\\ &\quad\le \int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\left[1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\right]dx\\ &\quad\le \frac{p}{m-1}\int_0^b \frac{z(x)}{z'(x)}w\frac{z'(x)}{z^m(x)}F^{p-1}|g(x)|\,dx.\end{aligned}$$

By Hölder's inequality

$$\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx \le \frac{\lambda p}{m-1}\left(\int_0^b w\frac{z'(x)}{z^m(x)}F^p dx\right)^{\frac{1}{q}}\left(\int_0^b w\frac{z^{p-m}(x)}{[z'(x)]^{\frac{p}{q}}}|g(x)|^p\,dx\right)^{\frac{1}{p}}$$

and thus on simplification, we have

$$\int_0^b w(x)\frac{z'(x)}{z^m(x)}F^p(x)\,dx \le \left(\frac{\lambda p}{m-1}\right)^p\int_0^b w\frac{z^{p-m}(x)}{[z'(x)]^{\frac{p}{q}}}|g(x)|^p\,dx.$$

This proves the theorem. □

Theorem 2.4. *Let* p,m,f,z,z',w *and* r *be as in Theorem 2.1. Let*

$$1-\frac{1}{m-1}\frac{z(x)}{z'(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{z(x)}{z'(x)}\frac{r'(x)}{r(x)}\ge\frac{1}{\lambda}>0.$$

Let $a \in (0, \infty)$ *be fixed and set*

$$F(x) := r(x) \int_{\frac{x}{2}}^{x} \frac{z'(t) f(t)}{z(t) r(t)} dt, \qquad x \in (0, \infty).$$

Then the inequality

$$\int_0^b w(x) \frac{z'(x)}{z^m(x)} F^p(x)\, dx \le \left(\frac{\lambda p}{m-1} \right)^p \int_0^b w(x) \frac{[z(x)]^{p-m}}{[z'(x)]^{\frac{p}{q}}} |g(x)|^p\, dx \tag{2.4}$$

holds where

$$g(x) = r(x) \left(\frac{r(x) z'(x) f(x)}{z(x)} - \frac{1}{2} \frac{r(\frac{x}{2}) z'(\frac{x}{2}) f(\frac{x}{2})}{z(\frac{x}{2})} \right).$$

Proof. This is similar to the proof of the Theorem 2.3. □

Theorem 2.5. *Let* $p > q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $m < 1$ *be real numbers. Let* $z(x)$ *be differentiable in* $(0, \infty)$ *with* $z'(x) > 0$ *and* $z(0_+) > 0$, *let* $w(x)$ *and* $r(x)$ *be positive and absolutely continuous functions on* $(0, \infty), n$ *and let* $f : [0, \infty) \to [0, \infty)$ *be integrable so that*

$$1 + \frac{1}{m-1} \frac{z(x)}{z'(x)} \frac{w'(x)}{w(x)} - \frac{p}{q} \frac{1}{m-1} \frac{z(x)}{z'(x)} \frac{r'(x)}{r(x)} \ge \frac{1}{\lambda} > 0$$

a.e. for some $\lambda > 0$. *Let* $b \in (0, \infty)$ *be fixed and set*

$$F(x) := \frac{1}{r(x)} \int_x^b \frac{r(t) z'(t) f(t)}{z(t)} dt, \qquad x \in (0, \infty).$$

Then the following inequality

$$\int_a^b w(x) \frac{z'(x)}{z^m(x)} F^{\frac{p}{q}}(x)\, dx \le \left(\frac{\lambda p}{m-1} \right)^p \int_a^b w(x) \frac{z'(x)}{z^m(x)} f^{\frac{p}{q}}(x) dx \tag{2.5}$$

holds for all $0 \le a \le b$.

Proof. This is similar to the proof of the Theorem 2.1. □

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