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# EXISTENCE OF POSITIVE SOLUTIONS FOR NON LOCAL $p$-LAPLACIAN THERMISTOR PROBLEMS ON TIME SCALES

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ABSTRACT. We make use of the Guo-Krasnoselskii fixed point theorem on cones to prove existence of positive solutions to a non local $p$-Laplacian boundary value problem on time scales arising in many applications.

*Key words and phrases:* Time scales, $p$-Laplacian, Positive solutions, Existence.

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## 1. INTRODUCTION

The purpose of this paper is to prove the existence of positive solutions for the following non local $p$-Laplacian dynamic equation on a time scale $\mathbb{T}$:

$$-\left(\phi_p(u^{\triangle}(t))\right)^{\nabla} = \frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\,\nabla\tau\right)^k}, \quad \forall t \in (0,T)_{\mathbb{T}} = \mathbb{T}\,, \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} \phi_p(u^{\triangle}(0)) - \beta\phi_p(u^{\triangle}(\eta)) = 0, \quad 0 < \eta < T, \\ u(T) - \beta u(\eta) = 0, \end{aligned} \tag{1.2}$$

where $\phi_p(\cdot)$ is the $p$-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$ with $q$ the Hölder conjugate of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The function

$$f : (0,T)_{\mathbb{T}} \to \mathbb{R}^{+*} \quad \text{is assumed to be continuous} \tag{H1}$$

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($\mathbb{R}^{+*}$ denotes the positive real numbers); $\lambda$ is a dimensionless parameter that can be identified with the square of the applied potential difference at the ends of a conductor; $f(u)$ is the temperature dependent resistivity of the conductor; $\beta$ is a transfer coefficient supposed to verify $0 < \beta < 1$. Different values for $p$ and $k$ are connected with a variety of applications for both $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. When $k > 1$, equation (1.1) represents the thermo-electric flow in a conductor [20]. In the particular case $p = k = 2$, (1.1) has been used to describe the operation of thermistors, fuse wires, electric arcs and fluorescent lights [11, 12, 18, 19]. For $k = 1$, equation (1.1) models the phenomena associated with the occurrence of shear bands (i) in metals being deformed under high strain rates [6, 7], (ii) in the theory of gravitational equilibrium of polytropic stars [17], (iii) in the investigation of the fully turbulent behavior of real flows, using invariant measures for the Euler equation [10], (iv) in modelling aggregation of cells via interaction with a chemical substance (chemotaxis) [22].

The theory of dynamic equations on time scales (or, more generally, measure chains) was introduced in 1988 by Stefan Hilger in his PhD thesis (see [14, 15]). The theory presents a structure where, once a result is established for a general time scale, then special cases include a result for differential equations (obtained by taking the time scale to be the real numbers) and a result for difference equations (obtained by taking the time scale to be the integers). A great deal of work has been done since 1988, unifying and extending the theories of differential and difference equations, and many results are now available in the general setting of time scales – see [1, 2, 3, 4, 8, 9] and the references therein. We point out, however, that results concerning $p$-Laplacian problems on time scales are scarce [21]. In this paper we prove the existence of positive solutions to the problem (1.1)-(1.2) on a general time scale $\mathbb{T}$.

## 2. PRELIMINARIES

Our main tool to prove the existence of positive solutions (Theorem 3.5) is the Guo-Krasnoselskii fixed point theorem on cones.

**Theorem 2.1** (Guo-Krasnoselskii fixed point theorem on cones [13, 16])**.** *Let $X$ be a Banach space and $K \subset E$ be a cone in $X$. Assume that $\Omega_1$ and $\Omega_2$ are bounded open subsets of $K$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and that $G : K \to K$ is a completely continuous operator such that*

(i) *either $\|Gw\| \leq \|w\|$, $w \in \partial\Omega_1$, and $\|Gw\| \geq \|w\|$, $w \in \partial\Omega_2$; or*
(ii) *$\|Gw\| \geq \|w\|$, $w \in \partial\Omega_1$, and $\|Gw\| \leq \|w\|$, $w \in \partial\Omega_2$.*

*Then, $G$ has a fixed point in $\overline{\Omega}_2 \backslash \Omega_1$.*

Using the properties of $f$ on a bounded set $(0, T)_{\mathbb{T}}$, we construct an operator (an integral equation) whose fixed points are solutions to the problem (1.1)-(1.2).

Now we introduce some basic concepts of time scales that are needed in the sequel. For deeper details, the reader can see, for instance, [1, 5, 8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The *forward jump* operator $\sigma$ and the *backward jump* operator $\rho$, both from $\mathbb{T}$ to $\mathbb{T}$, are defined in [14]:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}\,.$$

A point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t,\ \rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right scattered minimum $m$, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum $M$, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ (assume $t$ is not left-scattered if $t = \sup \mathbb{T}$), then the delta derivative of $f$ at the point $t$ is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the

property that for each $\epsilon > 0$ there is a neighborhood $U$ of $t$ such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq |\sigma(t) - s|, \quad \text{for all } s \in U\,.$$

Similarly, for $t \in \mathbb{T}$ (assume $t$ is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of $f$ at the point $t$ is defined to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood $U$ of $t$ such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all } s \in U\,.$$

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

A function $f$ is left-dense continuous ($ld$-continuous) if $f$ is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. Let $f$ be $ld$-continuous. If $F^{\nabla}(t) = f(t)$, then the nabla integral is defined by

$$\int_a^b f(t)\nabla t = F(b) - F(a)\,;$$

if $F^{\Delta}(t) = f(t)$, then the delta integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a)\,.$$

In the remainder of this article $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with $0 \in \mathbb{T}_k$, $T \in \mathbb{T}^k$; $E = \mathbb{C}_{ld}([0,T],\mathbb{R})$, which is a Banach space with the maximum norm $\|u\| = \max_{[0,T]_{\mathbb{T}}} |u(t)|$.

## 3. MAIN RESULTS

By a positive solution of (1.1)-(1.2) we understand a function $u(t)$ which is positive on $(0,T)_{\mathbb{T}}$ and satisfies (1.1) and (1.2).

**Lemma 3.1.** *Assume that hypothesis (H1) is satisfied. Then, $u(t)$ is a solution of* (1.1)-(1.2) *if and only if $u(t) \in E$ is solution of the integral equation*

$$u(t) = -\int_0^t \phi_q\left(g(s)\right) \triangle s + B,$$

*where*

$$g(s) = \int_0^s \lambda h(u(r))\nabla r - A,$$

$$A = \phi_p(u^{\triangle}(0)) = -\frac{\lambda\beta}{1-\beta}\int_0^{\eta} h(u(r))\nabla r,$$

$$h(u(t)) = \frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\,\nabla\tau\right)^k},$$

$$B = u(0) = \frac{1}{1-\beta}\left\{\int_0^T \phi_q(g(s))\triangle s - \beta\int_0^{\eta}\phi_q(g(s))\triangle s\right\}.$$

*Proof.* We begin by proving necessity. Integrating the equation (1.1) we have

$$\phi_p(u^{\triangle}(s)) = \phi_p(u^{\triangle}(0)) - \int_0^s \lambda h(u(r))\nabla r.$$

On the other hand, by the boundary condition (1.2)

$$\phi_p(u^{\triangle}(0)) = \beta\phi_p(u^{\triangle}(\eta)) = \beta\left(\phi_p(u^{\triangle}(0)) - \int_0^{\eta}\lambda h(u(r))\nabla r\right).$$

Then,

$$A = \phi_p(u^{\triangle}(0)) = \frac{-\lambda\beta}{1-\beta}\int_0^{\eta} h(u(r))\nabla r.$$

It follows that

$$u^{\triangle}(s) = \phi_q\left(-\lambda\int_0^s h(u(r))\nabla r + A\right) = -\phi_q(g(s)).$$

Integrating the last equation, we obtain

$$u(t) = u(0) - \int_0^t \phi_q(g(s))\triangle s. \tag{3.1}$$

Moreover, by (3.1) and the boundary condition (1.2), we have

$$\begin{aligned} u(0) &= u(T) + \int_0^T \phi_q(g(s))\triangle s \\ &= \beta u(\eta) + \int_0^T \phi_q(g(s))\triangle s \\ &= \beta\left(u(0) - \int_0^{\eta} \phi_q(g(s))\triangle s\right) + \int_0^T \phi_q(g(s))\triangle s. \end{aligned}$$

Then,

$$u(0) = B = \frac{1}{1-\beta}\left(-\beta\int_0^{\eta}\phi_q(g(s))\triangle s + \int_0^T \phi_q(g(s))\triangle s\right).$$

Sufficiency follows by a simple calculation, taking the delta derivative of $u(t)$. □

**Lemma 3.2.** *Suppose (H1) holds. Then, a solution* $u$ *of* (1.1)-(1.2) *satisfies* $u(t) \geq 0$ *for all* $t \in (0,T)_{\mathbb{T}}$.

*Proof.* We have $A = \frac{-\lambda\beta}{1-\beta}\int_0^{\eta} h(u(r))\nabla r \leq 0$. Then, $g(s) = \lambda\int_0^s h(u(r)) - A \geq 0$. It follows that $\phi_p(g(s)) \geq 0$. Since $0 < \beta < 1$, we also have

$$\begin{aligned} u(0) = B &= \frac{1}{1-\beta}\left\{\int_0^T \phi_q(g(s))\triangle s - \beta\int_0^{\eta}\phi_q(g(s))\triangle s\right\} \\ &\geq \frac{1}{1-\beta}\left\{\beta\int_0^T \phi_q(g(s))\triangle s - \beta\int_0^{\eta}\phi_q(g(s))\triangle s\right\} \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} u(T) &= u(0) - \int_0^T \phi_q(g(s))\triangle s \\ &= \frac{-\beta}{1-\beta}\int_0^{\eta}\phi_q(g(s))\triangle s + \frac{1}{1-\beta}\int_0^T \phi_q(g(s))\triangle s - \int_0^T \phi_q(g(s))\triangle s \\ &= \frac{-\beta}{1-\beta}\int_0^{\eta}\phi_q(g(s))\triangle s + \frac{\beta}{1-\beta}\int_0^T \phi_q(g(s))\triangle s \\ &= \frac{\beta}{1-\beta}\left\{\int_0^T \phi_q(g(s))\triangle s - \int_0^{\eta}\phi_q(g(s))\triangle s\right\} \\ &\geq 0. \end{aligned}$$

If $t \in (0,T)_{\mathbb{T}}$,

$$\begin{aligned} u(t) &= u(0) - \int_0^t \phi_q(g(s)) \triangle s \\ &\geq - \int_0^T \phi_q(g(s)) \triangle s + u(0) = u(T) \geq 0\,. \end{aligned}$$

□

**Lemma 3.3.** *If (H1) holds, then* $u(T) \geq \rho u(0)$, *where* $\rho = \beta \frac{T-\eta}{T-\beta\eta} \geq 0$.

*Proof.* We have $\phi_p(u^{\triangle}(s)) = \phi_p(u^{\triangle}(0)) - \int_0^s \lambda h(u(r)) \nabla r \leq 0$. Since $A = \phi_p(u^{\triangle}(0)) \leq 0$, then $u^{\triangle} \leq 0$. This means that $\|u\| = u(0)$, $\inf_{t\in(0,T)_{\mathbb{T}}} u(t) = u(T)$. Moreover, $\phi_p(u^{\triangle}(s))$ is non increasing which implies, with the monotonicity of $\phi_p$, that $u^{\triangle}$ is a non increasing function on $(0,T)_{\mathbb{T}}$. It follows from the concavity of $u(t)$ that each point on the chord between $(0,u(0))$ and $(T,u(T))$ is below the graph of $u(t)$. We have

$$u(T) \geq u(0) + T\frac{u(T)-u(\eta)}{T-\eta}.$$

Alternatively,

$$Tu(\eta) - \eta u(T) \geq (T-\eta)u(0).$$

Using the boundary condition (1.2), it follows that

$$\left(\frac{T}{\beta} - \eta\right) u(T) \geq (T-\eta)u(0).$$

Then,

$$u(T) \geq \beta\frac{T-\eta}{T-\beta\eta}u(0).$$

□

In order to apply Theorem 2.1, we define the cone $K$ by

$$K = \left\{ u \in E, u \text{ is concave on } (0,T)_{\mathbb{T}} \text{ and } \inf_{t\in(0,T)_{\mathbb{T}}} u(t) \geq \rho\|u\| \right\}.$$

It is easy to see that (1.1)-(1.2) has a solution $u = u(t)$ if and only if $u$ is a fixed point of the operator $G : K \to E$ defined by

$$Gu(t) = -\int_0^t \phi_q\left(g(s)\right) \triangle s + B, \tag{3.2}$$

where $g$ and $B$ are defined as in Lemma 3.1.

**Lemma 3.4.** *Let* $G$ *be defined by* (3.2). *Then,*

(i) $G(K) \subseteq K$;
(ii) $G : K \to K$ *is completely continuous.*

*Proof.* Condition $(i)$ holds from previous lemmas. We now prove $(ii)$. Suppose that $D \subseteq K$ is a bounded set. Let $u \in D$. We have:

$$\begin{aligned} |Gu(t)| &= \left| -\int_0^t \phi_q\left(g(s)\right) \triangle s + B \right| \\ &= \left| -\int_0^t \phi_q\left( \int_0^s \frac{\lambda f(u(r))}{(\int_0^T f(u(\tau))\,\nabla\tau)^k} \nabla r - A \right) \triangle s + B \right| \end{aligned}$$

$$\leq \int_0^T \phi_q \left( \int_0^s \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D})^k} \nabla r - A \right) \triangle s + |B|,$$

$$\begin{aligned} |A| &= \left| \frac{\lambda\beta}{1-\beta} \int_0^\eta h(u(r)) \nabla r \right| \\ &= \left| \frac{\lambda\beta}{1-\beta} \int_0^\eta \frac{f(u(r))}{(\int_0^T f(u(\tau))\, \nabla r)^k} \nabla r \right| \\ &\leq \frac{\lambda\beta}{1-\beta} \frac{\sup_{u\in D} f(u)}{(T \inf_{u\in D})^k}\ \eta. \end{aligned}$$

In the same way, we have

$$\begin{aligned} |B| &\leq \frac{1}{1-\beta} \int_0^T \phi_q(g(s)) \triangle s \\ &\leq \frac{1}{1-\beta} \int_0^T \phi_q \left( \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D})^k} \left( s + \frac{\beta}{1-\beta} \eta \right) \right) \triangle s\,. \end{aligned}$$

It follows that

$$|Gu(t)| \leq \int_0^T \phi_q \left( \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D})^k} \left( s + \frac{\beta\eta}{1-\beta} \right) \right) \triangle s + |B|.$$

As a consequence, we get

$$\begin{aligned} \|Gu\| &\leq \frac{2-\beta}{1-\beta} \int_0^T \phi_q \left( \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D})^k} \left( s + \frac{\beta\eta}{1-\beta} \right) \right) \\ &\leq \frac{2}{1-\beta} \phi_q \left( \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D})^k} \right) \int_0^T \phi_q \left( s + \frac{\beta\eta}{1-\beta} \right) \triangle s\,. \end{aligned}$$

We conclude that $G(D)$ is bounded. Item $(ii)$ follows by a standard application of the Arzela-Ascoli and Lebesgue dominated theorems. □

**Theorem 3.5** (Existence result on cones)**.** *Suppose that (H1) holds. Assume furthermore that there exist two positive numbers $a$ and $b$ such that*

(H2) $$\max_{0\leq u\leq a} f(u) \leq \phi_p(aA_1),$$

(H3) $$\min_{0\leq u\leq b} f(u) \geq \phi_p(bB_1),$$

*where*

$$A_1 = \frac{1-\beta}{T(2-\beta)} \phi_p \left( \frac{1}{(T \inf_{0\leq u\leq a} f(u))^k} \left( T + \frac{\beta\eta}{1-\beta} \right) \right)$$

*and*

$$B_1 = \frac{1-\beta}{\beta(T-\eta)} \phi_p(\eta) \phi_p \left( \frac{\lambda}{\left(T \sup_{0\leq u\leq b} f(u)\right)^k} \right).$$

*Then, there exists $0 < \lambda_* < 1$ such that the non local $p$-Laplacian problem* (1.1)-(1.2) *has at least one positive solution $\overline{u}$, $a \leq \overline{u} \leq b$, for any $\lambda \in (0, \lambda_*)$.*

*Proof.* Let $\Omega_r = \{u \in K, \|u\| \leq r\}$, $\partial\Omega_r = \{u \in K, \|u\| = r\}$. If $u \in \partial\Omega_a$, then $0 \leq u \leq a$, $t \in (0,T)_{\mathbb{T}}$. This implies $f(u(t)) \leq \max_{0\leq u\leq a} f(u) \leq \phi_p(aA)$. We can write that

$$\begin{aligned}\|Gu\| &\leq \int_0^T \phi_q(g(s))\triangle s + B \\ &\leq \int_0^T \phi_q\left(\int_0^s \frac{\lambda f(u(r))}{(\int_0^T f(u(\tau))\,\nabla\tau)^k}\nabla r - A\right)\triangle s + B\,,\end{aligned}$$

$$|A| = \frac{\lambda\beta}{1-\beta}\int_0^\eta \frac{f(u(r))}{(\int_0^T f(u(\tau))\,\nabla\tau)^k}\nabla r \leq \frac{\lambda\beta}{1-\beta}\frac{(aA_1)^{p-1}}{(T\inf_{0\leq u\leq a} f(u))^k}\eta\,,$$

$$g(s) \leq \frac{\lambda(aA_1)^{p-1}}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right).$$

Then,

$$\begin{aligned}\int_0^T \phi_q(g(s))\triangle s &\leq \phi_q\left(\frac{\lambda(aA_1)^{p-1}}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right)T \\ &= aA_1T\phi_q\left(\frac{\lambda}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right).\end{aligned}$$

Moreover,

$$\begin{aligned}B &= \frac{1}{1-\beta}\left(\int_0^T \phi_q(g(s))\triangle s - \beta\int_0^\eta \phi_q(g(s))\triangle s\right) \\ &\leq \frac{1}{1-\beta}\left(\int_0^T \phi_q(g(s))\triangle s\right) \\ &\leq aA_1\frac{T}{1-\beta}\phi_q\left(\frac{\lambda}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right).\end{aligned}$$

For $A_1$ as in the statement of the theorem, it follows that

$$\begin{aligned}\|Gu\| &\leq aA_1T\frac{2-\beta}{1-\beta}\phi_q\left(\frac{\lambda}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right) \\ &\leq \phi_q(\lambda)aA_1T\frac{2-\beta}{1-\beta}\phi_q\left(\frac{1}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right) \\ &\leq \phi_q(\lambda_*)aA_1T\frac{2-\beta}{1-\beta}\phi_q\left(\frac{1}{(T\inf_{0\leq u\leq a} f(u))^k}\left(T + \frac{\beta\eta}{1-\beta}\right)\right) \\ &\leq \phi_q(\lambda_*)a \\ &\leq a = \|u\|.\end{aligned}$$

If $u \in \partial\Omega_b$, we have

$$\begin{aligned}\|Gu\| &\geq -\int_0^T \phi_q\left(g(s)\right) \triangle s + B \\ &\geq -\int_0^T \phi_q\left(g(s)\right) \triangle s + \frac{1}{1-\beta}\int_0^T \phi_q\left(g(s)\right) \triangle s - \frac{\beta}{1-\beta}\int_0^\eta \phi_q\left(g(s)\right) \triangle s \\ &\geq \frac{\beta}{1-\beta}\int_0^T \phi_q\left(g(s)\right) \triangle s - \frac{\beta}{1-\beta}\int_0^\eta \phi_q\left(g(s)\right) \triangle s \\ &\geq \frac{\beta}{1-\beta}\int_\eta^T \phi_q\left(g(s)\right) \triangle s.\end{aligned}$$

Since $A \leq 0$, we have

$$\begin{aligned}g(s) &= \lambda\int_0^s h(u(r))\nabla r - A \geq \lambda\int_0^s h(u(r))\nabla r \\ &\geq \lambda\int_0^s \frac{f(u)}{(T\sup_{0\leq u\leq b} f(u))^k} \\ &\geq \lambda\frac{(bB_1)^{p-1}}{(T\sup_{0\leq u\leq b})^k}s.\end{aligned}$$

Using the fact that $\phi_q$ is nondecreasing we get

$$\begin{aligned}\phi_q(g(s)) &\geq \phi_q\left(\lambda\frac{(bB_1)^{p-1}}{(T\sup_{0\leq u\leq b})^k}s\right) \\ &\geq bB_1\phi_q\left(\frac{\lambda}{(T\sup f(u))^k}\right)\phi_q(s).\end{aligned}$$

Then, using the expression of $B_1$,

$$\begin{aligned}\|Gu\| &\geq \frac{\beta}{1-\beta}bB_1\phi_q\left(\frac{\lambda}{(T\sup f(u))^k}\right)\int_\eta^T \phi_q(s)\triangle s \\ &\geq bB_1\frac{\beta}{1-\beta}\phi_q\left(\frac{\lambda}{(T\sup f(u))^k}\right)\phi_q(\eta)(T-\eta) \\ &\geq b = \|u\|.\end{aligned}$$

As a consequence of Lemma 3.4 and Theorem 2.1, $G$ has a fixed point theorem $\overline{u}$ such that $a \leq \overline{u} \leq b$. □

## 4. AN EXAMPLE

We consider a function $f$ which arises with the negative coefficient thermistor (NTC-thermistor). For this example the electrical resistivity decreases with the temperature.

**Corollary 4.1.** *Assume (H1) holds. If*

$$f_0 = \lim_{u\to 0}\frac{f(u)}{\phi_p(u)} = 0\,, \quad f_\infty = \lim_{u\to\infty}\frac{f(u)}{\phi_p(u)} = +\infty,$$

*or*

$$f_0 = +\infty\,, \quad f_\infty = 0\,,$$

*then problem* (1.1)-(1.2) *has at least one positive solution.*

*Proof.* If $f_0 = 0$ then $\forall\ A_1 > 0\ \exists\ a$ such that $f(u) \leq (A_1 u)^{p-1}$, $0 \leq u \leq a$. Similarly as above, we can prove that $\|Gu\| \leq \|u\|$, $\forall\ u \in \partial\Omega_a$. On the other hand, if $f_\infty = +\infty$, then $\forall\ B_1 > 0$, $\exists\ b > 0$ such that $f(u) \geq (B_1 u)^{p-1}$, $u \geq b$. As in the proof of Theorem 3.5, we have $\|Gu\| \geq \|u\|$, $\forall\ u \in \partial\Omega_b$. By Theorem 2.1, $G$ has a fixed point. □

For the NTC-thermistor, the dependence of the resistivity to the temperature can be expressed by

$$f(s) = \frac{1}{(1+s)^k}\,, \quad k \geq 2\,. \tag{4.1}$$

For $p = 2$, we have

$$f_0 = \lim_{u\to 0} \frac{f(u)}{\phi_p(u)} = +\infty\,, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{\phi_p(u)} = 0\,.$$

It follows from Corollary 4.1 that the boundary value problem (1.1)-(1.2) with $p = 2$ and $f$ as in (4.1) has at least one positive solution.

## REFERENCES


[1] R.P. AGARWAL AND M. BOHNER, Basic calculus on time scales and some of its applications, *Result. Math.*, **35** (1999), 3–22.

[2] R.P. AGARWAL, M. BOHNER, D. O'REGAN AND A. PETERSON, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.*, **142** (2002), 1–26.

[3] R.P. AGARWAL, M. BOHNER AND P. WONG, Sturm-Liouville eigenvalue problem on time scales, *Appl. Math. Comput.*, **99** (1999), 153–166.

[4] R.P. AGARWAL AND D. O'REGAN, Nonlinear boundary value problems on time scales, *Nonlinear Anal.*, **44** (2001), 527–535.

[5] F.M. ATICI AND G.Sh. GUSEINOV, On Green's functions and positive solutions for boundary-value problems on time scales, *J. Comput. Appl. Math.*, **141** (2002), 75–99.

[6] J.W. BEBERNES AND A.A. LACEY, Global existence and finite-time blow-up for a class of non-local parabolic problems, *Adv. Diff. Eqns.*, **2** (1997), 927–953.

[7] J. W. BEBERNES, C. LI AND P. TALAGA, Single-point blow-up for non-local parabolic problems, *Physica D*, **134** (1999), 48–60.

[8] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales—An Introduction with Applications*, Birkhäuser, Boston, 2001.

[9] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Cambridge, MA, 2003.

[10] E. CAGLIOTI, P-L. LIONS, C. MARCHIORO AND M. PULVIRENTI, A special class of stationary flows for two-dimensinal Euler equations: a statistical mechanics description, *Comm. Math. Phys.*, **143** (1992), 501–525.

[11] A. EL HACHIMI AND M.R. SIDI AMMI, Semidiscretization for a nonlocal parabolic problem, *Int. J. Math. Math. Sci.*, **2005**(10) (2005), 1655–1664.

[12] A. EL HACHIMI, M.R. SIDI AMMI AND D.F.M. TORRES, A dual mesh method for a nonlocal thermistor problem, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **2** (2006), Paper 058, 10 pp. (electronic). [ONLINE: `http://www.emis.de/journals/SIGMA/2006/Paper058/index.html`]

[13] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic press, Boston, 1988.

[14] S. HILGER, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56.

[15] S. HILGER, Differential and difference calculus—unified!, *Nonlinear Anal.*, **30**(5) (1997), 2683–2694.

[16] M.A. KRASNOSELSKII, *Positive Solutions of Operator Equations*, Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron, P. Noordhoff Ltd. Groningen, The Netherlands, 1964.

[17] A. KRZYWICKI AND T. NADZIEJA, Some results concerning the Poisson-Boltzmann equation, *Zastos. Mat.*, **21**(2) (1991), 265–272.

[18] A.A. LACEY, Thermal runaway in a non-local problem modelling ohmic heating. Part I: Model derivation and some special cases, *Euro. J. Appl. Math.*, **6** (1995), 127–144.

[19] A.A. LACEY, Thermal runaway in a non-local problem modelling ohmic heating. Part II: General proof of blow-up and asymptotics of runaway, *Euro. J. Appl. Math.*, **6** (1995), 201–224.

[20] A.A. LACEY, Diffusion models with blow-up, *J. Comp. Appl. Math.*, **97** (1998), 39–49.

[21] D.-B. WANG, Existence, multiplicity and infinite solvability of positive solutions for $p$-Laplacian dynamic equations on time scales, *Electron. J. Diff. Eqns.*, **2006**(96) (2006), 1–10.

[22] G. WOLANSKY, A critical parabolic estimate and application to non-local equations arising in chemotaxis, *Appl. Anal.*, **66** (1997), 291–321.