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A GENERAL OPTIMAL INEQUALITY FOR ARBITRARY RIEMANNIAN SUBMANIFOLDS

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ABSTRACT. One of the most fundamental problems in submanifold theory is to establish simple relationships between intrinsic and extrinsic invariants of the submanifolds (cf. [6]). A general optimal inequality for submanifolds in Riemannian manifolds of constant sectional curvature was obtained in an earlier article [5]. In this article we extend this inequality to a general optimal inequality for arbitrary Riemannian submanifolds in an arbitrary Riemannian manifold. This new inequality involves only the δ-invariants, the squared mean curvature of the submanifolds and the maximum sectional curvature of the ambient manifold. Several applications of this new general inequality are also presented.

Key words and phrases: δ-invariants, Inequality, Riemannian submanifold, Squared mean curvature, Sectional curvature.

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1. INTRODUCTION

According to the celebrated embedding theorem of J.F. Nash [23], every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension. The Nash theorem was established in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. However, as observed by M. Gromov [18], this hope had not been materialized. The main reason for this is due to the lack of controls of the extrinsic properties of the submanifolds by the known intrinsic data.

In order to overcome the difficulty mentioned above, the author introduced in [4, 5] some new types of Riemannian invariants, denoted by $\delta(n_1, \ldots, n_k)$. Moreover, he was able to establish in [5] an optimal general inequality for submanifolds in real space forms which involves his δ-invariants and the main extrinsic invariant; namely, the squared mean curvature. Such inequality provides prima controls on the most important extrinsic curvature invariant by the initial intrinsic data of the Riemannian submanifolds in real space forms. As an application,

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he was able to discover new intrinsic spectral properties of homogeneous spaces via Nash's theorem. Such results extend a well-known theorem of Nagano [22]. Since then the δ-invariant and the inequality established in [5] have been further investigated by many geometers (see for instance, [2, 8, 9, 10, 11, 14, 12, 13, 15, 17, 20, 21, 24, 25, 26, 27, 28, 29, 30]). Recently, the δ-invariants have also been applied to general relativity theory as well as to affine geometry (see for instance, [7, 16, 19]).

In this article we use the same idea introduced in the earlier article [5] to extend the inequality mentioned above to a more general optimal inequality for an arbitrary Riemannian submanifold in an arbitrary Riemannian manifold.

Our general inequality involves the δ-invariant, the squared mean curvature of the Riemannian submanifold and the maximum of the sectional curvature function of the ambient Riemannian manifold (restricted to plane sections of the tangent space of the submanifold at a point on the submanifold). More precisely, we prove in Section 3 that, for any n-dimensional submanifold M in a Riemannian m-manifold $\tilde{M}^m$, we have the following general optimal inequality:

$$\delta(n_1,\dots,n_k) \le c(n_1,\dots,n_k)H^2 + b(n_1,\dots,n_k)\max\tilde{K} \tag{1.1}$$

for any k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, where $\max\tilde{K}(p)$ denotes the maximum of the sectional curvature function of $\tilde{M}^m$ restricted to 2-plane sections of the tangent space T_pM of M at p. (see Section 3 for details). (When $k=0$, inequality (1.1) can be found in B. Suceavă's article [27]).

In the last section we provide several immediate applications of inequality (1.1). In particular, by applying our inequality we conclude that if M is a Riemannian n-manifold with $\delta(n_1,\dots,n_k)>0$ at some point in M for some k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, then M admits no minimal isometric immersion into any Riemannian manifold with non-positive sectional curvature. In this section, we also apply inequality (1.1) to derive two inequalities for submanifolds in Sasakian space forms. In fact, many inequalities for submanifolds in various space forms obtained by various people can also be derived directly from inequality (1.1).

2. Preliminaries

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame

$$e_1,\dots,e_n,e_{n+1},\dots,e_m$$

in $\tilde{M}^m$ such that, restricted to M, the vectors $e_1,\dots,e_n$ are tangent to M and hence $e_{n+1},\dots,e_m$ are normal to M. Let $K(e_i\wedge e_j)$ and $\tilde{K}(e_i\wedge e_j)$ denote respectively the sectional curvatures of M and $\tilde{M}^m$ of the plane section spanned by e_i and e_j.

For the submanifold M in $\tilde{M}^m$ we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [3])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X\xi \tag{2.2}$$

for any vector fields X,Y tangent to M and vector field ξ normal to M, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let $\{h_{ij}^r\}$, $i,j = 1,\dots,n$; $r = n+1,\dots,m$, denote the coefficients of the second fundamental form h with respect to $e_1,\dots,e_n,e_{n+1},\dots,e_m$. Then we have

$$h_{ij}^r = \langle h(e_i,e_j),e_r\rangle = \langle A_{e_r}e_i,e_j\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\,\mathrm{trace}\,h = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i), \tag{2.3}$$

where $\{e_1,\dots,e_n\}$ is a local orthonormal frame of the tangent bundle TM of M. The squared mean curvature is then given by $H^2 = \left\langle \overrightarrow{H},\overrightarrow{H}\right\rangle$. A submanifold M is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Then the *equation of Gauss* is given by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle\,, \tag{2.4}$$

for vectors X,Y,Z,W tangent to M.

For any orthonormal basis $e_1,\dots,e_n$ of the tangent space T_pM, the scalar curvature τ of M at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i\wedge e_j), \tag{2.5}$$

where $K(e_i\wedge e_j)$ denotes the sectional curvature of the plane section spanned by e_i and e_j.

Let L be a subspace of T_pM of dimension $r\geq 1$ and $\{e_1,\dots,e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha\wedge e_\beta), \quad 1\leq \alpha,\beta\leq r. \tag{2.6}$$

When $r=1$, we have $\tau(L)=0$.

For integers $k\geq 0$ and $n\geq 2$, let us denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1,\dots,n_k)$ of integers ≥ 2 which satisfies

$$n_1 < n \;\text{ and }\; n_1+\cdots+n_k\leq n.$$

Let $\mathcal{S}(n)$ be the union $\cup_{k\geq 0}\,\mathcal{S}(n,k)$.

For any $(n_1,\dots,n_k)\in\mathcal{S}(n)$, the Riemannian invariants $\delta(n_1,\dots,n_k)$ introduced in [5] are defined by

$$\delta(n_1,\dots,n_k)(p) = \tau(p) - \inf\{\tau(L_1)+\cdots+\tau(L_k)\}, \tag{2.7}$$

where $L_1,\dots,L_k$ run over all k mutually orthogonal subspaces of T_pM with $\dim L_j = n_j$, $j = 1,\dots,k$.

We recall the following general algebraic lemma from [4] for later use.

Lemma 2.1. *Let $a_1,\dots,a_n,\eta$ be $n+1$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\eta + \sum_{i=1}^{n} a_i^2\right).$$

Then $2a_1a_2\geq\eta$, with equality holding if and only if we have

$$a_1+a_2 = a_3 = \cdots = a_n.$$

3. A GENERAL OPTIMAL INEQUALITY

For each $(n_1,\dots,n_k)\in\mathcal{S}(n)$, let $c(n_1,\dots,n_k)$ and $b(n_1,\dots,n_k)$ be the positive numbers given by

$$c(n_1,\dots,n_k)=\frac{n^2(n+k-1-\sum_{j=1}^k n_j)}{2(n+k-\sum_{j=1}^k n_j)}, \tag{3.1}$$

$$b(n_1,\dots,n_k)=\frac{1}{2}(n(n-1)-\sum_{j=1}^k n_j(n_j-1)). \tag{3.2}$$

For an arbitrary Riemannian submanifold we have the following general optimal inequality.

Theorem 3.1. *Let M be an n-dimensional submanifold of an arbitrary Riemannian m-manifold $\tilde M^m$. Then, for each point $p\in M$ and for each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, we have the following inequality:*

$$\delta(n_1,\dots,n_k)(p)\le c(n_1,\dots,n_k)H^2(p)+b(n_1,\dots,n_k)\max\tilde K(p), \tag{3.3}$$

where $\max\tilde K(p)$ denotes the maximum of the sectional curvature function of $\tilde M^m$ restricted to 2-plane sections of the tangent space T_pM of M at p.

The equality case of inequality (3.3) *holds at a point $p\in M$ if and only the following two conditions hold:*

(a) There exists an orthonormal basis $e_1,\dots,e_m$ at p, such that the shape operators of M in $\tilde M^m$ at p take the following form :

$$A_{e_r}=\begin{bmatrix} \begin{matrix} A_1^r & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & A_k^r\end{matrix} & 0\\ 0 & \mu_r I\end{bmatrix},\quad r=n+1,\dots,m, \tag{3.4}$$

where I is an identity matrix and each A_j^r is a symmetric $n_j\times n_j$ submatrix such that

$$\text{trace}\,(A_1^r)=\cdots=\text{trace}\,(A_k^r)=\mu_r. \tag{3.5}$$

(b) For any k mutual orthogonal subspaces $L_1,\dots,L_k$ of T_pM which satisfy

$$\delta(n_1,\dots,n_k)=\tau-\sum_{j=1}^k\tau(L_j)$$

at p, we have

$$\tilde K(e_{\alpha_i},e_{\alpha_j})=\max\tilde K(p) \tag{3.6}$$

for any $\alpha_i\in\Delta_i,\alpha_j\in\Delta_j$ with $i\neq j$, where

$$\Delta_1=\{1,\dots,n_1\},$$
$$\dots$$
$$\Delta_k=\{n_1+\cdots+n_{k-1}+1,\dots,n_1+\cdots+n_k\}.$$

Proof. Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde M^m$ and p be a point in M. Then the equation of Gauss implies that at p we have

$$2\tau(p)=n^2H^2-||h||^2+2\tilde\tau(T_pM), \tag{3.7}$$

where $||h||^2$ is the squared norm of the second fundamental form h and $\tilde{\tau}(T_pM)$ is the scalar curvature of the ambient space $\tilde{M}^m$ corresponding to the subspace $T_pM \subset T_p\tilde{M}^m$, i.e.

$$\tilde{\tau}(T_pM) = \sum_{i<j} \tilde{K}(e_i, e_j)$$

for an orthonormal basis $e_1, \ldots, e_n$ of T_pM.

Let us put

$$\eta = 2\tau(p) - \frac{n^2(n+k-1-\sum n_j)}{n+k-\sum n_j} H^2 - 2\tilde{\tau}(T_pM). \tag{3.8}$$

Then we obtain from (3.7) and (3.8) that

$$n^2H^2 = \gamma\left(\eta + ||h||^2\right), \quad \gamma = n + k - \sum n_j. \tag{3.9}$$

At p, let us choose an orthonormal basis $e_1, \ldots, e_m$ such that $e_{\alpha_i} \in L_i$ for each $\alpha_i \in \Delta_i$. Moreover, we choose the normal vector e_{n+1} to be in the direction of the mean curvature vector at p (When the mean curvature vanishes at p, e_{n+1} can be chosen to be any unit normal vector at p). Then (3.9) yields

$$\left(\sum_{A=1}^{n} a_A\right)^2 = \gamma\left[\eta + \sum_{A \neq B}(h_{AB}^{n+1})^2 + \sum_{A=1}^{n}(a_A)^2 + \sum_{r=n+2}^{m}\sum_{A,B=1}^{n}(h_{AB}^r)^2\right], \tag{3.10}$$

where $a_A = h_{AA}^{n+1}$ with $1 \leq A, B \leq n$. Equation (3.10) is equivalent to

$$\begin{aligned}\left(\sum_{i=1}^{\gamma+1} \bar{a}_i\right)^2 = \gamma\Bigg[\eta + \sum_{i=1}^{\gamma+1}(\bar{a}_i)^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ - \sum_{1\leq\alpha_1\neq\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} - \sum_{\alpha_2\neq\beta_2} a_{\alpha_2}a_{\beta_2} - \cdots \sum_{\alpha_k\neq\beta_k} a_{\alpha_k}a_{\beta_k}\Bigg],\end{aligned} \tag{3.11}$$

where $\alpha_2, \beta_2 \in \Delta_2, \ \ldots, \ \alpha_k, \beta_k \in \Delta_k$ and

$$\bar{a}_1 = a_1, \ \bar{a}_2 = a_2 + \cdots + a_{n_1},$$

$$\bar{a}_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \tag{3.12}$$

$$\cdots \tag{3.13}$$

$$\bar{a}_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1\cdots+n_k}, \tag{3.14}$$

$$\bar{a}_{k+2} = a_{n_1\cdots+n_k+1}, \ldots, \bar{a}_{\gamma+1} = a_n. \tag{3.15}$$

By applying Lemma 2.1 to (3.11) we obtain

$$\begin{aligned}\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \\ \geq \frac{\eta}{2} + \sum_{A<B}(h_{AB}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=1}^{n}(h_{AB}^r)^2,\end{aligned} \tag{3.16}$$

where $\alpha_i, \beta_i \in \Delta_i, \ i = 1, \ldots, k$.

On the other hand, equation (2.6) and the equation of Gauss imply that, for each $j \in \{1, \ldots, k\}$, we have

$$\tau(L_j) = \sum_{r=n+1}^{m} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\alpha_j} h^r_{\beta_j\beta_j} - (h^r_{\alpha_j\beta_j})^2\right) + \tilde{\tau}(L_j), \quad \alpha_j, \beta_j \in \Delta_j. \tag{3.17}$$

where $\tilde{\tau}(L_j)$ is the scalar curvature $\tilde{M}^m$ asociated with $L_j \subset T_p$.

Let us put $\Delta = \Delta_1 \cup \cdots \cup \Delta_k$ and $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k)$. Then we obtain by combining (3.8), (3.16) and (3.17) that

$$\begin{aligned} \tau(L_1) + \cdots + \tau(L_k) &\geq \frac{\eta}{2} + \frac{1}{2} \sum_{r=n+1}^{m} \sum_{(\alpha,\beta)\notin\Delta^2} (h^r_{\alpha\beta})^2 \\ &\quad + \frac{1}{2} \sum_{r=n+2}^{m} \sum_{j=1}^{k} \left(\sum_{\alpha_j\in\Delta_j} h^r_{\alpha_j\alpha_j} \right)^2 + \sum_{j=1}^{k} \tilde{\tau}(L_j) \\ &\geq \frac{\eta}{2} + \sum_{j=1}^{k} \tilde{\tau}(L_j) \\ &= \tau - \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 - \left(\tilde{\tau}(T_pM) - \sum_{j=1}^{k} \tilde{\tau}(L_j) \right). \end{aligned} \tag{3.18}$$

Therefore, by (2.7) and (3.18), we obtain

$$\tau - \sum_{j=1}^{k} \tau(L_j) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \tilde{\tau}(T_pM) - \sum_{j=1}^{k} \tilde{\tau}(L_j), \tag{3.19}$$

which implies that

$$\delta(n_1, \ldots, n_k) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \tilde{\delta}^M(n_1, \ldots, n_k), \tag{3.20}$$

where

$$\tilde{\delta}^M(n_1, \ldots, n_k) := \tilde{\tau}(T_pM) - \inf\{\tilde{\tau}(\tilde{L}_1) + \cdots + \tilde{\tau}(\tilde{L}_k)\} \tag{3.21}$$

with $\tilde{L}_1, \ldots, \tilde{L}_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim \tilde{L}_j = n_j$; $j = 1, \ldots, k$. Clearly, inequality (3.21) implies inequality (3.3).

It is easy to see that the equality case of (3.3) holds at the point p if and only if the following two conditions hold:

(i) The inequalities in (3.16) and (3.18) are actually equalities;

(ii) For any k mutual orthogonal subspaces $L_1, \ldots, L_k$ of T_pM which satisfy

$$\delta(n_1, \ldots, n_k) = \tau - \sum_{j=1}^{k} \tau(L_j) \tag{3.22}$$

at p, we have

$$\tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p) \tag{3.23}$$

for any $\alpha_i \in \Delta_i, \alpha_j \in \Delta_j$ with $i \neq j$.

It follows from Lemma 2.1, (3.16) and (3.18) that condition (i) holds if and only if there exists an orthonormal basis $e_1, \ldots, e_m$ at p, such that the shape operators of M in $\tilde{M}^m$ at p satisfy conditions (3.4) and (3.5).

The converse can be easily verified. □

4. Some Applications

The following results follow immediately from Theorem 3.1

Theorem 4.1. *Let M be an n-dimensional submanifold of the complex projective m-space $CP^m(4\epsilon)$ of constant holomorphic sectional curvature 4ϵ (or the quaternionic projective m-space $QP^m(4\epsilon)$ of quaternionic sectional curvature 4ϵ). Then we have*

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + 4b(n_1, \ldots, n_k)\epsilon \tag{4.1}$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

Theorem 4.2. *Let M be an n-dimensional submanifold of the complex hyperbolic m-space $CH^m(4\epsilon)$ of constant holomorphic sectional curvature $4c$ (or the quaternionic hyperbolic m-space $QH^m(4\epsilon)$ of quaternionic sectional curvature 4ϵ). Then we have*

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\epsilon \tag{4.2}$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

Theorem 4.3. *Let $\tilde{M}^m$ be a Riemannian manifold whose sectional curvature function is bounded above by ϵ. If M is a Riemannian n-manifold such that*

$$\delta(n_1, \ldots, n_k)(p) > \frac{1}{2}\left(n(n-1) - \sum n_j(n_j - 1)\right)\epsilon$$

for some k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ at some point $p \in M$, then M admits no minimal isometric immersion in $\tilde{M}^m$.

In particular, we have the following non-existence result.

Corollary 4.4. *If M is a Riemannian n-manifold with*

$$\delta(n_1, \ldots, n_k) > 0$$

at some point in M for some k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, then M admits no minimal isometric immersion into any Riemannian m-manifold $\tilde{M}^m$ with non-positive sectional curvature, regardless of codimension.

A $(2m+1)$-dimensional manifold is called *almost contact* if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4.3}$$

where I is the identity endomorphism. It is well-known that

$$\phi\xi = 0, \quad \eta \circ \phi = 0.$$

Moreover, the endomorphism ϕ has rank $2m$.

An almost contact manifold $(\tilde{M}, \phi, \xi, \eta)$ is called an *almost contact metric manifold* if it admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4.4}$$

for vector fields X, Y tangent to $\tilde{M}$. Setting $Y = \xi$ we have immediately that

$$\eta(X) = g(X, \xi).$$

By a *contact manifold* we mean a $(2m+1)$-manifold $\tilde{M}$ together with a global 1-form η satisfying

$$\eta \wedge (d\eta)^m \neq 0$$

on M. If η of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is a contact form and if η satisfies

$$d\eta(X, Y) = g(X, \phi Y)$$

for all vectors X, Y tangent to $\tilde{M}$, then $\tilde{M}$ is called a *contact metric manifold*.

A contact metric manifold is called K-*contact* if its characteristic vector field ξ is a Killing vector field. It is well-known that a K-contact metric $(2n+1)$-manifold satisfies

$$\nabla_X \xi = -\phi X, \quad \tilde{K}(X, \xi) = 1 \tag{4.5}$$

for $X \in \ker \eta$, where $\tilde{K}$ denotes the sectional curvature on M.

A K-contact manifold is called *Sasakian* if we have

$$N_\phi + 2d\eta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor associated to ϕ. A plane section σ in $T_p\tilde{M}^{2m+1}$ of a Sasakian manifold $\tilde{M}^{2m+1}$ is called ϕ-section if it is spanned by X and $\phi(X)$, where X is a unit tangent vector orthogonal to ξ. The sectional curvature with respect to a ϕ-section σ is called a ϕ-sectional curvature. If a Sasakian manifold has constant ϕ-sectional curvature, it is called a Sasakian space form.

An n-dimensional submanifold M^n of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\tilde{M}^{2m+1}(c)$ if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\phi(TM^n) \subset T^\perp M^n$, which means that M^n is an anti-invariant submanifold of $\tilde{M}^{2m+1}(c)$

It is well-known that the Riemannian curvature tensor of a Sasakian space form $\tilde{M}^{2m+1}(\epsilon)$ of constant ϕ-sectional curvature ϵ is given by [1]:

$$\begin{aligned} \tilde{R}(X,Y)Z = {} & \frac{\epsilon+3}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle Y) \\ & + \frac{\epsilon-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X,Z\rangle \eta(Y)\xi \\ & \qquad - \langle Y,Z\rangle \eta(X)\xi + \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y - 2\langle \phi X, Y\rangle \phi Z) \end{aligned} \tag{4.6}$$

for X, Y, Z tangent to $\tilde{M}^{2m+1}(\epsilon)$. Hence if $\epsilon \geq 1$, the sectional curvature function $\tilde{K}$ of $\tilde{M}^{2m+1}(\epsilon)$ satisfies

$$\frac{\epsilon+3}{4} \leq \tilde{K}(X,Y) \leq \epsilon \tag{4.7}$$

for $X, Y \in \ker \eta$; if $\epsilon < 1$, the inequalities are reversed.

From Theorem 3.1 and these sectional curvature properties (4.5) and (4.7) of Sasakian space forms, we obtain the following results for arbitrary Riemannian submanifolds in Sasakian space forms.

Corollary 4.5. *If M is an n-dimensional submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant ϕ-sectional curvature $\epsilon \geq 1$, then, for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k) H^2(p) + b(n_1, \ldots, n_k)\epsilon. \tag{4.8}$$

Corollary 4.6. *If M is an n-dimensional submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant ϕ-sectional curvature $\epsilon < 1$, then, for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k) H^2(p) + b(n_1, \ldots, n_k). \tag{4.9}$$

Corollary 4.7. *If M is an n-dimensional C-totally real submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant ϕ-sectional curvature $\epsilon \leq 1$, then, for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\frac{\epsilon + 3}{4}. \tag{4.10}$$

Corollary 4.7 has been obtained in [13].

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