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ATOMIC DECOMPOSITIONS FOR WEAK HARDY SPACES W$\mathcal{Q}_p$ AND W$\mathcal{D}_p$

YANBO REN AND DEWU YANG

DEPARTMENT OF MATHEMATICS AND PHYSICS
HENAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
LUOYANG 471003, P.R. CHINA
ryb7945@sina.com.cn

dewuyang0930@163.com

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ABSTRACT. In this paper some necessary and sufficient conditions for new forms of atomic decompositions of weak martingale Hardy spaces w$\mathcal{Q}_p$ and w$\mathcal{D}_p$ are obtained.

Key words and phrases: Martingale, Weak Hardy space, Atomic decomposition.

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1. INTRODUCTION AND PRELIMINARIES

It is well known that the method of atomic decompositions plays an important role in martingale theory, such as in the study of martingale inequalities and of the duality theorems for martingale Hardy spaces. Many theorems can be proved more easily through its use. The technique of stopping times used in the case of one-parameter is usually unsuitable for the case of multi-parameters, but the method of atomic decompositions can deal with them in the same manner. F.Weisz [6] gave some atomic decomposition theorems on martingale spaces and proved many important martingale inequalities and the duality theorems for martingale Hardy spaces with the help of atomic decompositions. Hou and Ren [3] obtained some weak types of martingale inequalities through the use of atomic decompositions.

In this paper we will establish some new atomic decompositions for weak martingale Hardy spaces w$\mathcal{Q}_p$ and w$\mathcal{D}_p$, and give some necessary and sufficient conditions.

Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space, and $(\Sigma_n)_{n\geq 0}$ a non-decreasing sequence of sub-σ-algebras of Σ such that $\Sigma = \sigma\left(\bigcup_{n\geq 0}\Sigma_n\right)$. The expectation operator and the conditional expectation operators relative to Σ_n are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. For a martingale $f = (f_n)_{n\geq 0}$ relative to $(\Omega, \Sigma, \mathbb{P}, (\Sigma_n)_{n\geq 0})$, define $df_i = f_i - f_{i-1}$ ($i \geq 0$, with convention $df_0 = 0$) and

$$f_n^* = \sup_{0\leq i\leq n} |f_i|, \qquad f^* = f_\infty^* = \sup_{n\geq 0} |f_n|,$$

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$$S_n(f) = \left(\sum_{i=0}^{n} |df_i|^2\right)^{\frac{1}{2}}, \qquad S(f) = \left(\sum_{i=0}^{\infty} |df_i|^2\right)^{\frac{1}{2}}.$$

Let $0 < p < \infty$. The space consisting of all measurable functions f for which

$$\|f\|_{\mathrm{w}L_p} =: \sup_{y>0} y\mathbb{P}(|f| > y)^{\frac{1}{p}} < \infty$$

is called a weak L_p-space and denoted by $\mathrm{w}L_p$. We set $\mathrm{w}L_\infty = L_\infty$. It is well-known that $\|\cdot\|_{\mathrm{w}L_p}$ is a quasi-norm on $\mathrm{w}L_p$ and $L_p \subset \mathrm{w}L_p$ since $\|f\|_{\mathrm{w}L_p} \le \|f\|_{L_p}$. Denote by Λ the collection of all sequences $(\lambda_n)_{n\ge 0}$ of non-decreasing, non-negative and adapted functions and set $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. If $0 < p < \infty$, we define the weak Hardy spaces as follows:

$$\mathrm{w}\mathcal{Q}_p = \{f = (f_n)_{n\ge 0} : \exists (\lambda_n)_{n\ge 0} \in \Lambda, \text{ s.t. } S_n(f) \le \lambda_{n-1}, \lambda_\infty \in \mathrm{w}L_p\},$$
$$\|f\|_{\mathrm{w}\mathcal{Q}_p} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\mathrm{w}L_p};$$
$$\mathrm{w}\mathcal{D}_p = \{f = (f_n)_{n\ge 0} : \exists (\lambda_n)_{n\ge 0} \in \Lambda, \text{ s.t. } |f_n| \le \lambda_{n-1}, \lambda_\infty \in \mathrm{w}L_p\},$$
$$\|f\|_{\mathrm{w}\mathcal{D}_p} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\mathrm{w}L_p}.$$

Remark 1. Similar to martingale Hardy spaces $\mathcal{Q}_p$ and $\mathcal{D}_p$ (see F.Weisz [6]), we can prove that "inf" in the definitions of $\|\cdot\|_{\mathrm{w}\mathcal{Q}_p}$ and $\|\cdot\|_{\mathrm{w}\mathcal{D}_p}$ is attainable. That is, there exist $(\lambda_n^{(1)})_{n\ge 0}$ and $(\lambda_n^{(2)})_{n\ge 0}$ such that $\|f\|_{\mathrm{w}\mathcal{Q}_p} = \|\lambda_\infty^{(1)}\|_{\mathrm{w}L_p}$ and $\|f\|_{\mathrm{w}\mathcal{D}_p} = \|\lambda_\infty^{(2)}\|_{\mathrm{w}L_p}$, which are called the optimal control of $S(f)$ and f, respectively.

Definition 1.1 ([6]). Let $0 < p < \infty$. A measurable function a is called a $(2, p, \infty)$ atom (or $(3, p, \infty)$ atom) if there exists a stopping time ν (ν is called the stopping time associated with a) such that

(i) $a_n = \mathbb{E}_n a = 0$ if $\nu \ge n$,
(ii) $\|S(a)\|_\infty \le \mathbb{P}(\nu \ne \infty)^{-\frac{1}{p}}$ (or (ii)′ $\|a^*\|_\infty \le \mathbb{P}(\nu \ne \infty)^{-\frac{1}{p}}$).

Throughout this paper, we denote the set of integers and the set of non-negative integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use C_p to denote constants which depend only on p and may denote different constants at different occurrences.

2. MAIN RESULTS AND PROOFS

Atomic decompositions for weak martingale Hardy spaces $\mathrm{w}\mathcal{Q}_p$ and $\mathrm{w}\mathcal{D}_p$ have been established in [3]. In this section, we give them new forms of atomic decompositions, which are closely connected with weak type martingale inequalities.

Theorem 2.1. *Let $0 < p < \infty$. Then the following statements are equivalent:*

(i) *There exists a constant $C_p > 0$ such that for each martingale $f = (f_n)_{\ge 0}$:*

$$\|f^*\|_{\mathrm{w}L_p} \le C_p \|f\|_{\mathrm{w}\mathcal{Q}_p};$$

(ii) *If $f = (f_n)_{n\ge 0} \in \mathrm{w}\mathcal{Q}_p$, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(3, p, \infty)$ atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that for all $n \in \mathbb{N}$:*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \tag{2.1}$$

and

$$\sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \le C_p \parallel f \parallel_{\mathrm{w}\mathcal{Q}_p}, \tag{2.2}$$

where $0 \leq \mu_k \leq A \cdot 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ *for some constant* A *and* ν_k *is the stopping time associated with* a^k*.*

Proof. (i)⇒(ii). Let $f = (f_n)_{\geq 0} \in \mathrm{w}\mathcal{Q}_p$. Then there exists an optimal control $(\lambda_n)_{n\geq 0}$ such that $S_n(f) \leq \lambda_{n-1}$. Consequently,

$$|f_n| \leq f^*_{n-1} + \lambda_{n-1}. \tag{2.3}$$

Define stopping times for all $k \in \mathbb{Z}$:

$$\nu_k = \inf\{n \geq 0 : f^*_n + \lambda_n > 2^k\}, \quad (\inf \emptyset = \infty).$$

The sequence of stopping times is obviously non-decreasing. Let $f^{\nu_k} = (f_{n\wedge\nu_k})_{n\geq 0}$ be the stopped martingale. Then

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) &= \sum_{k\in\mathbb{Z}} \left(\sum_{m=0}^{n} \chi(m \leq \nu_{k+1}) df_m - \sum_{m=0}^{n} \chi(m \leq \nu_k) df_m \right) \\
&= \sum_{m=0}^{n} \left(\sum_{k\in\mathbb{Z}} \chi(\nu_k < m \leq \nu_{k+1}) df_m \right) = f_n,
\end{aligned} \tag{2.4}$$

where $\chi(A)$ denotes the characteristic function of the set A. Now let

$$\mu_k = 2^k \cdot 3\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}, \quad a_n^k = \mu_k^{-1}(f_n^{\nu_{k+1}} - f_n^{\nu_k}), \quad (k \in \mathbb{Z}, n \in \mathbb{N}) \tag{2.5}$$

($a_n^k = 0$ if $\mu_k = 0$). It is clear that for a fixed $k \in \mathbb{Z}$, $(a_n^k)_{n\geq 0}$ is a martingale, and by (2.3) we have

$$\left|a_n^k\right| \leq \mu_k^{-1}(|f_n^{\nu_{k+1}}| + |f_n^{\nu_k}|) \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}. \tag{2.6}$$

Consequently, $(a_n^k)_{n\geq 0}$ is L_2-bounded and so there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k = a_n^k$, $n \geq 0$. It is clear that $a_n^k = 0$ if $n \leq \nu_k$ and by (2.6) we get $\|a^{k*}\|_\infty \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$. Therefore each a^k is a $(3, p, \infty)$ atom, (2.4) and (2.5) shows that f has a decomposition of the form (2.1) and $0 \leq \mu_k \leq A \cdot 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ with $A = 3$, respectively. By (i), we have

$$\begin{aligned}
2^{kp}\mathbb{P}\left(\nu_k < \infty\right) &= 2^{kp}\mathbb{P}(f^* + \lambda_\infty > 2^k) \\
&\leq 2^{kp}(\mathbb{P}(f^* > 2^{k-1}) + \mathbb{P}(\lambda_\infty > 2^{k-1})) \\
&\leq C_p \|f\|^p_{\mathrm{w}\mathcal{Q}_p},
\end{aligned}$$

which proves (2.2).

(ii)⇒(i). Let $f = (f_n)_{\geq 0} \in \mathrm{w}\mathcal{Q}_p$. Then f can be decomposed as in (ii) $f_n = \sum_{k\in\mathbb{Z}} \mu_k a_n^k$ of $(3, p, \infty)$ atoms such that (2.2) holds. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$ and let

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k = \sum_{k=-\infty}^{j-1} \mu_k a^k + \sum_{k=j}^{\infty} \mu_k a^k =: g + h.$$

It follows from the sublinearity of maximal operators that we have $f^* \leq g^* + h^*$, so

$$\mathbb{P}(f^* > 2y) \leq \mathbb{P}(g^* > y) + \mathbb{P}(h^* > y).$$

For $0 < p < \infty$, choose q so that $\max(1, p) < q < \infty$. By (ii) and the fact that $a^{k*} = 0$ on the set $(\nu_k = \infty)$, we have

$$\begin{aligned}
\|g^*\|_q &\leq \sum_{k=-\infty}^{j-1} \mu_k \|a^{k*}\|_q = \sum_{k=-\infty}^{j-1} \mu_k \|a^{k*}\chi(\nu_k \neq \infty)\|_q \\
&\leq \sum_{k=-\infty}^{j-1} A \cdot 2^{k(1-\frac{p}{q})} 2^{\frac{kp}{q}} \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{q}} \\
&\leq C_p \sum_{k=-\infty}^{j-1} A \cdot 2^{k(1-\frac{p}{q})} \|f\|_{\mathrm{w}\mathcal{Q}_p}^{\frac{p}{q}} \\
&\leq C_p y^{1-\frac{p}{q}} \|f\|_{\mathrm{w}\mathcal{Q}_p}^{\frac{p}{q}}.
\end{aligned}$$

It follows that

$$\mathbb{P}(g^* > y) \leq y^{-q}\mathbb{E}[g^{*q}] \leq C_p y^{-p} \|f\|_{\mathrm{w}\mathcal{Q}_p}^p. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{P}\left(h^* > y\right) \leq \mathbb{P}(h^* > 0) &\leq \sum_{k=j}^{\infty} \mathbb{P}(a^{k*} > 0) \\
&\leq \sum_{k=j}^{\infty} \mathbb{P}(\nu_k \neq \infty) \\
&\leq \sum_{k=j}^{\infty} 2^{-kp} \cdot 2^{kp} \mathbb{P}(\nu_k \neq \infty) \qquad (2.8) \\
&\leq C_p y^{-p} \|f\|_{\mathrm{w}\mathcal{Q}_p}^p.
\end{aligned}$$

Combining (2.7) with (2.8), we get $\mathbb{P}(f^* > y) \leq C_p y^{-p} \|f\|_{\mathrm{w}\mathcal{Q}_p}^p$. Hence

$$\|f^*\|_{\mathrm{w}L_p} \leq C_p \|f\|_{\mathrm{w}\mathcal{Q}_p}.$$

The proof is completed. □

Theorem 2.2. *Let $0 < p < \infty$. Then the following statements are equivalent:*

(i) *There exists a constant $C_p > 0$ such that for each martingale $f = (f_n)_{\geq 0}$:*

$$\|S(f)\|_{\mathrm{w}L_p} \leq C_p \|f\|_{\mathrm{w}\mathcal{D}_p};$$

(ii) *If $f = (f_n)_{n\geq 0} \in \mathrm{w}\mathcal{D}_p$, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(2, p, \infty)$ atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that for all $n \in \mathbb{N}$:*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \tag{2.9}$$

and

$$\sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \leq C_p \left\|f\right\|_{\mathrm{w}\mathcal{D}_p}, \tag{2.10}$$

where $0 \leq \mu_k \leq A \cdot 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ for some constant A and ν_k is the stopping time associated with a^k.

Proof. (i)⇒(ii). Let $f = (f_n)_{\geq 0} \in \mathrm{w}\mathcal{D}_p$. Then there exists an optimal control $(\lambda_n)_{n\geq 0}$ such that $|f_n| \leq \lambda_{n-1}$. Consequently,

$$S_n(f) = \left(\sum_{i=0}^{n-1} |df_i|^2 + |df_n|^2\right)^{\frac{1}{2}} \leq S_{n-1}(f) + 2\lambda_{n-1}. \tag{2.11}$$

Define stopping times for all $k \in \mathbb{Z}$:

$$\nu_k = \inf\{n \geq 0 : S_n(f) + 2\lambda_n > 2^k\}, \quad (\inf \emptyset = \infty),$$

and a_n^k and μ_k are as in the proof of Theorem 2.1. Then by (2.11) we have

$$S(a^k) \leq \mu_k^{-1}(S(f^{\nu_{k+1}}) + S(f^{\nu_k})) \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}.$$

Thus $\left\|S(a^k)\right\|_\infty \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$ and there exists an a^k such that $\mathbb{E}_n a^k = a_n^k$, $n \geq 0$. It is clear that a^k is a $(2, p, \infty)$ atom. Similar to the proof of Theorem 2.1, we can prove (2.9) and (2.10).

The proof of the implication (ii)⇒(i) is similar to that of Theorem 2.1.

The proof is completed. □

Remark 2. The two inequalities in (i) of Theorems 2.1 and 2.2 were obtained in [3]. Here we establish the relation between atomic decompositions of weak martingale Hardy spaces and martingale inequalities.

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