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IMPROVED $GA-$ CONVEXITY INEQUALITIES

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ABSTRACT. We consider a class of algebraic inequalities for functions of n variables depending on parameters that generalise the case of $GA-$convex functions. The functions in this class are $GA-$convex only in a subdomain of definition yet the inequality for $GA-$convexity still holds on the whole domain if suitable conditions are satisfied by the parameters. The method is elementary and allows us to give further extensions to a large class of functions.

As an application we show the validity of an n-dimensional generalization of a conjectured inequality related to a problem given at the 42nd IMO held at Washington DC (USA) in 2001.

Key words and phrases: Convex functions, $GA-$convexity, Symmetric functions, IMO competition.

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1. INTRODUCTION

The property of convexity of a given function $f : I \subset \mathbb{R} \to J \subset \mathbb{R}$ is one of the most powerful tools in establishing a wide range of analytic inequalities. As shown in [1] depending on which type of arithmetic (A) or geometric (G) mean we consider respectively on the domain and the co-domain of definition for f four classes of convex functions are distinguished. These are the AA-convexity (the usual convex functions), AG, GA or GG-convexity. Although a more general setting can be applied in the following, due to the geometric mean we shall assume throughout that $I, J \subset (0, \infty)$.

To be specific, $AG-$convex functions (or log-convex functions) are those functions $f : I \to (0, \infty)$ such that

$$x, y \in I,\ 0 \leq \alpha \leq 1 \Longrightarrow f\left((1-\alpha)\,x + \alpha y\right) \leq f^{1-\alpha}\left(x\right) f^{\alpha}\left(y\right). \tag{1.1}$$

This is equivalent that $\log f$ is convex.

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GG-convex functions (or multiplicatively convex functions) are those functions $f : I \to (0, \infty)$ such that

$$x, y \in I,\ 0 \le \alpha \le 1 \Longrightarrow f\left(x^{1-\alpha} y^{\alpha}\right) \le f^{1-\alpha}(x)\, f^{\alpha}(y). \tag{1.2}$$

Finally, $GA-$convex functions are those functions $f : I \to (0, \infty)$ such that

$$x, y \in I,\ 0 \le \alpha \le 1 \Longrightarrow f\left(x^{1-\alpha} y^{\alpha}\right) \le (1-\alpha) f(x) + \alpha f(y). \tag{1.3}$$

As can be checked rapidly every second order differentiable function satisfying

$$x^2 f''(x) + x f'(x) \ge 0 \ \text{ on its domain} \tag{1.4}$$

is $GA-$convex. In particular this is true if f is a convex and increasing function.

In [1] C.P. Niculescu discussed the beautiful class of inequalities, which arise from the notion of $GG-$convexity for functions. Clearly, a similar line of inquiry can be followed to analyse the class of inequalities arising by considering the remaining types of convexity such as GA and $AG-$convexity. In this paper we wish to extend the case of $GA-$convexity for second order differentiable functions for which inequality (1.4) is not satisfied in their entire domain of definition. Clearly to do so there must be some extra conditions imposed. Here we establish such conditions for the case when the functions depend also on extra parameters that obey given constraints. These cases lead us to a generalisation of the $GA-$convexity implicitly furnishing analytic inequalities, which cannot be established by the use of a direct method such as (1.4). Moreover, these results can in principle be extended to the other types of mean-convexity discussed above. In the first part we present the general result. As an illustration we establish an n-dimensional generalisation of an algebraic problem, which for the particular case of three variables, has appeared as a conjecture in relation to a proposed problem at the 42nd IMO held in Washington DC, USA 2001 [2]. The three variable conjecture has also appeared recently as proposal 10944 in the American Mathematical Monthly [3].

2. THE MAIN RESULT

Suppose that $f : (0, \infty) \to (0, \infty)$ is a second order differentiable function with $f''(x) \ge 0$ on its domain. Let $g : (0, \infty) \to (0, \infty)$, $g(x) = x^2 f''(x) + x f'(x)$. Suppose that there is $0 < r < 1$ with $g(r) = 0$ such that $g < 0$ on $(0, r)$ and $g \ge 0$ on (r, ∞). Further consider $h : (0, \infty)^n \to (0, \infty)$ defined by

$$h(x_1, \ldots, x_n) = \sum_{k=1}^{n} f(x_k) - n f(1),$$

for all $x_1, \ldots, x_n > 0$ with

$$\prod_{k=1}^{n} x_k = 1. \tag{2.1}$$

Finally assume that the components of the critical points of h subject to (2.1) can take at most two different values. That is there are $a \le b$ such that $\{x_1, \ldots, x_n\} = \{a, b\}$ at any critical point of components $(x_1, x_2, \cdots, x_n)$ of h.

Theorem 2.1. *If the above conditions are satisfied and, for every $k = 1, \ldots, n$, we have*

$$\lim_{x_k \to 0} h(x_1, \ldots, x_n) \ge 0 \tag{2.2}$$

then $h(x_1, \ldots, x_n) \ge 0$ for all $x_1, \ldots, x_n > 0$ with $\prod_{k=1}^{n} x_k = 1$.

Proof. First note that h is a continuous function defined on a bounded set from below therefore there is a value $m > -\infty$ such that $h \geq m$ for all $x_1, \ldots, x_n > 0$ with $\prod_{k=1}^{n} x_k = 1$. We shall show that $m \geq 0$ which will prove the theorem. Moreover, from (2.2), this is certainly true along the boundary of the domain, i.e. in the limit when $x_k \to 0$ for some $k = 1, \ldots, n$. Let

$$K = \left\{ (x_1, \ldots, x_n) \middle| x_1, \ldots, x_n > 0, \ \prod_{k=1}^{n} x_k = 1 \right\}.$$

To end the proof it remains to establish the assertion in the interior of K.

To do so we shall look at the extremum points of h. These are found from the critical points. By hypothesis their components can take at most two different values. That is

$$\frac{\partial h}{\partial x_i} = 0, i = 1, \ldots, n \Longrightarrow \left\{ x_1^0, \ldots, x_n^0 \right\} = \{a, b\}$$

at the critical points. Due to symmetry we can assume without restriction that $x_1^0 \leq \cdots \leq x_n^0$ and $a \leq b$. Therefore there exists $1 \leq q \leq n$ such that $x_1^0 = x_2^0 = \cdots = x_{q-1}^0 = a$ and $x_q^0 = x_{q+1}^0 = \cdots = x_n^0 = b$ (when $q = 1$ we use the convention that $x_0^0 = 0$). Note that (2.1) implies that $b \geq 1$. Also note that if $q = 1$ then there is nothing to prove as in this case the conclusion follows by applying condition (1.4) to the minimum point (or directly via (2.1)).

Next consider

$$h_1 (a, b) = (q - 1) f (a) + (n - q + 1) f (b) - nf (1) . \tag{2.3}$$

Note that via (2.1) we have that

$$a^{q-1} b^{n-q+1} = 1. \tag{2.4}$$

We shall show that $h_1 (a, b) \geq 0$ for all $a, b > 0$ satisfying (2.4). Via (2.4) this is equivalent to showing that

$$h_1 (b) = (q - 1) f \left(b^{\frac{n+1-q}{1-q}} \right) + (n - q + 1) f (b) - nf (1) \geq 0. \tag{2.5}$$

A simple calculation gives that $h_1' (b) = 0$ iff $h_1' (b) = h_1' \left(b^{\frac{n+1-q}{1-q}} \right) b^{\frac{n}{1-q}}$. Because $b \geq 1$ and f' is increasing the last equality is possible only when $b = 1$ in which case (2.5) becomes an equality. Moreover $h_1'' (1) = \frac{n}{q-1} (f'' (1) + f' (1)) \geq 0$ which follows from condition (1.4) applied to g at $x = 1$ and the fact that $r < 1$. This shows that $b_0 = 1$ is a minimum point for h_1 and that (2.5) is true at this point. Therefore it is true for all other points $b \geq 1$.

Finally, this establishes that the assertion is true at the minimum points of f and consequently this proves that the conclusion is true at all the interior points of the domain K. We have already verified it on the boundary of K so the proof is finished. □

3. AN APPLICATION

In a recent note [4] we gave a solution to a conjectured inequality in three positive variables which in turn is a generalisation of the 2nd problem given at the 42nd IMO held at Washington DC (USA) in 2001 [2]. The statement of the IMO problem was:

Problem 1. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1 \tag{3.1}$$

for all positive real numbers a, b and c.

At the end of the official IMO solution the author of the above proposed problem conjectured the following more general inequality:

Conjecture 3.1. *For any $a, b, c > 0$ and $\lambda \geq 8$, the following inequality holds*

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ca}}+\frac{c}{\sqrt{c^2+\lambda ab}} \geq \frac{3}{\sqrt{1+\lambda}}. \tag{3.2}$$

Using a direct calculatory method [4] we established the validity of (3.2). The same inequality has also been recently published as a proposal in *Amer. Math. Month.* [3]. Recently we learned about an algebraic solution to (3.2) that was obtained by Sava Grozdev (the team leader of the Bulgarian IMO team) [5]. However, his solution is very particular to the case of three positive numbers and so cannot be extended to the general case of n variables. In this direction we have proposed in [4] the following extension of (3.2) to the n-dimensional case.

Conjecture 3.2.

$$\sum_{i=1}^{n}\left((1+\lambda)\frac{x_i^{n-1}}{x_i^{n-1}+\lambda\prod_{k\neq i}x_k}\right)^{\frac{1}{n-1}} \geq n \tag{3.3}$$

for all $n \geq 1$, $x_i > 0$, $i = 1,\dots,n$ and any $\lambda \geq n^{n-1}-1$.

Inequality (3.3) has attracted interest (see [6]). In [6], Lagrange's method is used to show the validity of (3.3) but again the method is not amenable to further generalisation. Here we shall show that Conjecture 3.2 follows naturally from our main result above. However, before we do this it is useful to appreciate the strength of (3.3). First one can proceed as in [1] and exploit the property that the left hand side in (3.3) is homogeneous in the n-variables. Therefore with the natural transformation , $y_i = \frac{\prod_{k\neq i} x_i}{x_i^{n-1}}$, $i = 1,\dots,n$, one can reduce the problem to showing that

Theorem 3.3.

$$\sum_{i=1}^{n}\frac{1}{\sqrt[n-1]{1+\lambda y_i}} \geq \frac{n}{\sqrt[n-1]{1+\lambda}} \tag{3.4}$$

for all $n \geq 1$ and $y_i > 0$, $i = 1,\dots,n$ with the property $\prod_{i=1}^{n} y_i = 1$ and any $\lambda \geq n^{n-1}-1$.

There are some obvious suggestions to tackle (3.4). A naive approach would be to apply the $AM - GM$ inequality which would give that the AM of the left hand side of (3.3) is larger than $(1+\lambda)^{1/(n-1)}\prod_{i=1}^{n}(1+\lambda x_i)^{-1/(n(n-1))}$. However, the last expression is less than 1 rather than bigger to it (which is what we would have needed in order to obtain (3.4)) as can be easily checked by applying once more the $AM - GM$ inequality. Direct use of convexity properties does not appear too inspired either. For example the function generating the general term of the left hand side in (3.4) is convex. Therefore Jensen's inequality yields that the left hand side in (3.4) is larger than $n\left(1+\left(\frac{\lambda}{n}\right)\sum_{i=1}^{n}x_i\right)^{-1/(n-1)}$. However, the $AM - GM$ inequality with $\prod_{i=1}^{n} y_i = 1$ yields that

$$n(1+\lambda)^{-\frac{1}{(n-1)}} \geq \left(1+\left(\frac{\lambda}{n}\right)\sum_{i=1}^{n}x_i\right)^{-\frac{1}{(n-1)}}$$

so (3.4) cannot be established in this simple way either.

Note that when $n = 1, 2$ (3.4) is trivial and for $n = 3$ the validity of (3.4) was established in [4, 5] as discussed above. In this note we shall establish the validity of inequality (3.4) in general.

Proof of Theorem 3.3. The cases $n = 1, 2$ are immediate and we leave them as an exercise for the reader to attempt. In the following we shall discuss the case when $n > 2$. For any $n > 1$

and $x, \lambda > 0$ let $f(x) = (1+\lambda x)^{-1/(n-1)}$. It is easy to see that f is a decreasing and convex function of $x > 0$ for any $\lambda > 0$. Indeed we have that:

$$f'(x) = -\lambda (n-1)^{-1} (1+\lambda x)^{-n/(n-1)} < 0 \tag{3.5}$$

$$f''(x) = \lambda^2 n (n-1)^{-2} (1+\lambda x)^{-(2n-1)/(n-1)} > 0 \tag{3.6}$$

for all $x > 0$ and for any $\lambda > 0,\ n > 1$. Furthermore it is easy to see that

$$x^2 f''(x) + x f'(x) = x\lambda \frac{(1+\lambda x)^{(1-2n)/(n-1)}}{\lambda^2} (1 + x\lambda - n). \tag{3.7}$$

Therefore on the interval $J = \left(\frac{n-1}{\lambda}, \infty\right)$ f is $GA-$*convex*. From the hypothesis we also have that

$$\prod_{i=1}^{n} y_i = 1. \tag{3.8}$$

Therefore (3.4) becomes

$$hh(y_1, y_2, \ldots, y_n) = \sum_{k=1}^{n} f(y_i) - nf(1) \geq 0 \tag{3.9}$$

for all $y_i > 0,\ i = 1, \ldots, n$ satisfying (3.9) and all $\lambda \geq n^{n-1} - 1$.

It is easy to see that the critical points of hh subject to condition (3.8) must satisfy the equalities $d(y_1) = d(y_2) = \cdots = d(y_n)$, where $d(y) = \frac{y}{(1+\lambda y)^{n/(n-1)}}$. Now d is strictly monotonous on each of J and $\mathbb{R} - J$ so we deduce that the critical points of hh in (3.9) can attain at most two different values, let us say a and b, $\ a \leq b$. Moreover, (3.8) gives $b \geq 1$. At this stage we see that, with the possible exception of condition (2.2), all the hypothesis of Theorem 2.1 are satisfied in our case and so the conclusion follows for all the interior points of the domain.

We still need to check the behaviour on the frontier of the domain, that is the behaviour of hh in (3.9) when $a \to 0$ or (equivalently) $b \to \infty$. Because $\lim\limits_{a\to 0} f(a) = 1$, $\lim\limits_{b\to\infty} f(b) = 0$ we have to check that $q - 1 \geq nf(1) = n(1+\lambda x)^{-1/(n-1)}$ which is obviously true owing to the condition that $\lambda \geq n^{n-1} - 1$. Equality takes place when $q = 2$ and $\lambda = n^{n-1} - 1$. This verifies also that hypothesis (2.2) holds in our case.

These facts then establish inequality (3.9) for all critical points $y_i > 0,\ i = 1, \ldots, n$ and $\lambda \geq n^{n-1} - 1$. Therefore the proof finishes by applying Theorem 2.1. $\square$

Theorem 3.4. *For any* $\alpha, \beta > 0$, $n \geq 1$ *with* $\beta \geq (n^{n-1} - 1)\alpha$ *we have the inequality*

$$\sum_{i=1}^{n} \left(\frac{x_i^{n-1}}{\alpha x_i^{n-1} + \beta \prod_{k\neq i} x_k} \right)^{\frac{1}{n-1}} \geq n(\alpha+\beta)^{-\frac{1}{n-1}}. \tag{3.10}$$

The proof follows easily from Theorem 2.1 in a similar manner as in the proof of Theorem 3.3. A significantly extended version of Theorem 3.3 can in fact be established.

Theorem 3.5.

$$\sum_{i=1}^{n} (1+\lambda y_i)^{-1/p} \geq n(1+\lambda)^{-1/p} \tag{3.11}$$

for all $n \geq 1$, $p \geq 1$, $y_i > 0$, $i = 1, \ldots, n$ such that $\prod_{i=1}^{n} y_i = 1$ and any $\lambda \geq n^p - 1$.

The proof of this general inequality is absolutely similar to that in Theorem 3.3. In fact it can be done almost *ad litteram* by replacing the exponent $(n-1)$ by p in the arguments used in the proof of Theorem 3.3.

Theorems 3.3 and 3.5 also imply the validity of the following dual form of (3.3).

Theorem 3.6.

$$\sum_{i=1}^{n}\left((\alpha+\lambda\beta)\frac{\prod_{k\neq i}x_k}{\prod_{k\neq i}x_k+\lambda x_i^{n-1}}\right)^{-\frac{1}{p}}\geq n \tag{3.12}$$

for all $n\geq 1$, $p\geq 1$, $x_i>0$, $i=1,\dots,n$, $\alpha,\beta>0$ *and any* $\lambda\geq n^p-1$.

Proof. (3.12) follows from (3.10) – (3.11) via the transformation $x_i\to 1/x_i$, $i=1,\dots,n$. □

Corollary 3.7. *If* $\alpha,\beta>0$ *with* $\beta\geq\left(n^{n-1}-1\right)\alpha$ *then*

$$\sum_{i=1}^{n}\left(\alpha+\beta x_i^n\right)^{-\frac{1}{n-1}}\geq n\left(\alpha+\beta\right)^{-\frac{1}{n-1}} \tag{3.13}$$

for all $n\geq 1$, $x_i>0$, $i=1,\dots,n$ *such that* $\prod_{i=1}^{n}x_i=1$.

Proof. In (3.13) multiply both the denominator and the numerator of each term from the left hand side by x_i, $i=1,\dots,n$, respectively. □

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