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OUTER γ-CONVEX FUNCTIONS ON A NORMED SPACE

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Abstract

For some given positive γ, a function f is called outer γ-convex if it satisfies the Jensen inequality $f(z_i) \le (1-\lambda_i)f(x_0) + \lambda_i f(x_1)$ for some $z_0 := x_0, z_1, ..., z_k := x_1 \in [x_0, x_1]$ satisfying $\|z_i - z_{i+1}\| \le \gamma$, where $\lambda_i := \|x_0 - z_i\|/\|x_0 - x_1\|, i = 1, 2, ..., k-1$. Though the Jensen inequality is only required to hold true at some points (although the location of these points is uncertain) on the segment $[x_0, x_1]$, such a function has many interesting properties similar to those of classical convex functions. Among others it is shown that, if the infimum limit of an outer γ-convex function attains $-\infty$ at some point then this propagates to other points, and under some assumptions, a function is outer γ-convex iff its epigraph is an outer γ-convex set.

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Contents

1. Introduction

Convex functions belong to the most important objects investigated in mathematical programming. They have many interesting properties, for example, if a convex function attains $-\infty$ at some point then it attains $-\infty$ at every relative interior point of the domain, all lower level sets are convex and a function is convex iff its epigraph is convex; see [8]. It is worth mentioning that all of them follow from a single algebraic condition, namely the satisfaction of the Jensen inequality

$$f(x_\lambda) \leq (1-\lambda)f(x_0) + \lambda f(x_1) \tag{1.1}$$
$$x_\lambda = (1-\lambda)x_0 + \lambda x_1, \quad \lambda \in [0,1]$$

everywhere on the segment connecting two arbitrary points of the domain. In a generalization of the classical convexity, for allowing small nonconvex blips, convexity is required to hold true between points, the distance between which is greater than some given positive real number, say, the roughness degree. Suppose D is a nonempty convex set in the normed linear space $(X, \|\cdot\|)$. According to Klötzler and Hartwig ([1]), a function $f : D \subset X \to \mathbb{R}$ is called roughly ρ-convex if the Jensen inequality (1.1) is satisfied *for all points* $x_\lambda \in [x_0, x_1] \subset D$ whenever $\|x_1 - x_0\| > \rho$, for some given $\rho > 0$. But the requirement of (1.1) *at all points* is sometimes too hard (see [7]). In the concept of Hu, Klee, and Larman [2], a function f is called δ-convex if (1.1) is fulfilled *at each point* $x_\lambda \in [x_0, x_1]$ with

$$\|x_\lambda - x_0\| \geq \frac{\delta}{2} \quad \text{and} \quad \|x_\lambda - x_1\| \geq \frac{\delta}{2}$$

for some given $\delta > 0$, which means that at least $\|x_1 - x_0\| \geq \delta$. According to H. X. Phu, for a fixed $\gamma > 0$, a function f is called γ-convex if

$$\|x_1 - x_0\| \geq \gamma \quad \text{implies} \quad f(x_0') + f(x_1') \leq f(x_0) + f(x_1)$$
$$\text{with} \quad x_i' \in [x_0, x_1], \|x_i - x_i'\| = \gamma, \quad i = 0, 1$$

([6]). It follows that f must fulfill the Jensen inequality (1.1) *at least at* x_0' *or* x_1'. In addition to this trend, γ-convexlikeness and outer γ-convexity were introduced respectively in [4] and [5] (and they are equivalent for lower semi-continuous functions). We recall that a function f is called outer γ-convex if (1.1) holds true *for some points*

$$z_0 :\, = x_0, z_1, \ldots, z_k :\, = x_1 \in [x_0, x_1] \quad \text{satisfying} \quad \|z_{i+1} - z_i\| \leq \gamma$$

(but the location of these points is *uncertain*). It was shown in [4] that, under some assumptions, a function is outer γ-convex (convex, respectively) iff the sum of this function and an arbitrary continuous linear functional always fulfills the property “each lower level set is outer γ-convex" (“each lower level set is convex", respectively) (see the definition of outer γ-convex sets in Section 2).

In this paper we show that although the demand “satisfying (1.1) at some points which are uncertain where" of outer γ-convexity is very weak it could conclude some more similar properties of classical convexity. In Section 2 some similar properties of classical convexity are given. Among others we get the nearest-point properties (Proposition 2.2). Some properties of outer γ-convex functions are given in Section 3. In particular, if the infimum limit of an outer γ-convex function attains $-\infty$ at some point then this propagates to other points (the so-called infection property) (Proposition 3.4). Finally, under some assumptions, Corollary 4.2 says that a function is outer γ-convex iff it its epigraph is outer γ-convex.

2. Outer γ-Convex Sets

Let $(X, \|\cdot\|)$ be a normed linear space and γ be a fixed positive real number. For any $x_0, x_1 \in X$ and $\lambda \in [0,1]$, we denote

$$\begin{aligned} x_\lambda &:= (1-\lambda)x_0 + \lambda x_1, \\ [x_0, x_1] &:= \{x_\lambda : 0 \le \lambda \le 1\}, \\ [x_0, x_1[&:= [x_0, x_1] \setminus \{x_1\}, \\ ]x_0, x_1[&:= [x_0, x_1[\setminus \{x_0\}. \end{aligned}$$

As usual, $B(x, r) := \{y \in X : \|x - y\| \le r\}$ denotes the closed ball with centre x and radius $r > 0$. Let us recall the notion of outer γ-convex sets ([5]). A set $M \subset X$ is said to be *outer γ-convex* if for all $x_0, x_1 \in M$, there exist $z_0 := x_0, z_1, \ldots, z_k := x_1 \in [x_0, x_1] \cap M$ such that

$$\|z_{i+1} - z_i\| \le \gamma \quad \text{for} \quad i = 0, 1, \ldots, k-1. \tag{2.1}$$

Obviously, every convex set is outer γ-convex for all $\gamma > 0$. Conversely, if a closed set M is outer γ-convex for all $\gamma > 0$ then M must be convex. It follows directly from the following.

Proposition 2.1 ([5]). *Let $M \subset X$ be outer γ-convex, and let x_0 and x_1 belong to M. Then*

$$[x_0', x_1'[\subset [x_0, x_1] \setminus M \quad \textit{implies} \quad \|x_0' - x_1'\| < \gamma.$$

By virtue of this proposition, such a set M is called outer γ-convex because a segment connecting two points of M may contain at most gaps (i.e., subsegments outside M) whose length is smaller than γ.

For each $x \in X$, set $Mx := \{y^* \in M : \|x - y^*\| = \inf_{y \in M} \|x - y\|\}$.

Proposition 2.2. *Suppose that M is nonempty and outer γ-convex in X whose unit closed ball $B(0,1)$ is strictly convex. Then $\operatorname{diam} Mx \le \gamma$ for each $x \in X$.*

Proof. Assume the contrary that $\operatorname{diam} Mx > \gamma$. Then, there exist $x_0, x_1 \in Mx$ such that $\|x_0 - x_1\| > \gamma$. By the outer γ-convexity of M, there exists $z \in]x_0, x_1[\cap M$. The strict convexity of $B(0,1)$ implies $\|x - z\| < \max\{\|x - x_0\|, \|x - x_1\|\} = \|x - x_0\|$, which conflicts with $x_0 \in Mx$. □

Note that the assumption of the strict convexity of $B(0,1)$ is really needed. Moreover, the converse of Proposition 2.2 is false in case $\dim X \ge 2$. For example, the compact set

$$M := \{(x,y) \in \mathbb{R}^2 : x \in [-1,1], y \in [-1,1]\} \setminus \{(x,y) \in \mathbb{R}^2 : 0 < y < x\}$$

satisfies $\operatorname{diam} M(x,y) \le \gamma$ for all $(x,y) \in \mathbb{R}^2$, where $\gamma := 1$. But M is not outer γ-convex. As can be seen later, the converse of Proposition 2.2 holds true if $\dim X = 1$ and M is closed.

In view of Proposition 2.2, we get the following classical result which is a part of Motzkin's Theorem (see [9]).

Corollary 2.3. *Suppose that M is nonempty and convex in X whose unit closed ball $B(0,1)$ is strictly convex. Then, for each $x \in X$, if the set Mx is nonempty, it is a singleton.*

Proof. Since M is convex, it is outer γ-convex for all $\gamma > 0$. By Proposition 2.2, $\operatorname{diam} Mx \le \gamma$ for all $\gamma > 0$. It follows that $\operatorname{diam} Mx = 0$, i.e., Mx is a singleton. □

We recall that a set $M \subset X$ is *γ-convexlike* if $]x_0, x_1[\cap M \neq \emptyset$ holds true for all x_0, x_1 in M satisfying $\|x_0 - x_1\| > \gamma$ ([5]).

Clearly, each outer γ-convex set is γ-convexlike. In general the converse does not hold. The situation is quite different if M is closed.

Proposition 2.4 ([5]). *Suppose that M is closed. Then M is outer γ-convex iff it is γ-convexlike.*

Note that if $\dim X = 1$ and $\operatorname{diam} Mx \leq \gamma$ for each $x \in X$ then M is γ-convexlike (Indeed, if M were not γ-convexlike, i.e., there were $x_0, x_1 \in M$, $x_0 - x_1 > \gamma$ such that $]x_0, x_1[\cap M = \emptyset$, then $M\frac{x_0+x_1}{2} = \{x_0, x_1\}$ and therefore $\operatorname{diam} M\frac{x_0+x_1}{2} > \gamma$, a contradiction). Consequently, by Proposition 2.4, the converse of Proposition 2.2 holds true if $\dim X = 1$ and M is closed.

From Proposition 2.4 we have the following.

Proposition 2.5. *If M is outer γ-convex then $x_1, \dots, x_m \in M$ and*

$$\inf_{x \in \text{conv}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}} \|x_i - x\| > \gamma \quad \textit{for all } i = 1, \dots, m$$

and $m \geq 2$ imply that there exist $\lambda_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i x_i \in M$. If additionally M is closed, the converse is true.

Proof. Suppose that M is outer γ-convex. Then the above condition holds true for $m = 2$. It remains to prove that the above condition holds true for $m > 2$. The proof is by induction on m. Assume that the assertion holds for $m - 1$. Let $x_1, \dots, x_m \in M$ and

$$\inf_{x \in F_i} \|x_i - x\| > \gamma$$

for all $i = 1, \dots, m$, where $F_i : = \operatorname{conv}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}$. It implies that

$$\inf_{x \in F_i} \|x_i - x\| > \gamma$$

for all $i = 1, \dots, m-1$. Therefore, by the induction assumption, we conclude that

$$y : = \sum_{i=1}^{m-1} \lambda_i x_i \in M$$

with some $\lambda_i > 0$, $i = 1, \dots, m-1$ and $\sum_{i=1}^{m-1} \lambda_i = 1$. Since $\|y - x_m\| > \gamma$, there exists $\lambda_m \in \left]0, 1\right[$ such that $(1 - \lambda_m)y + \lambda_m x_m \in M$. Hence,

$$\sum_{i=1}^{m-1} \lambda_i (1 - \lambda_m) x_i + \lambda_m x_m \in M.$$

That is, the above condition always holds true.

Conversely, since the above condition holds true for $m = 2$, M is γ-convexlike. It follows from Proposition 2.4 that M is outer γ-convex. $\square$

3. Outer γ-Convex Functions

Suppose D is a nonempty convex set in the normed linear space $(X, \|\cdot\|)$. We recall that $f : D \subset X \to \mathbb{R}$ is *outer γ-convex* if for all distinct points $x_0, x_1 \in D$, there exist $z_0 := x_0, z_1, \dots, z_k := x_1 \in [x_0, x_1]$ satisfying (2.1) and

$$f(z_i) \leq (1 - \lambda_i) f(x_0) + \lambda_i f(x_1) \tag{3.1}$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|, \; i = 1, 2, \dots, k - 1$ (see [5]).

Clearly, a convex function is outer γ-convex for all $\gamma > 0$. Conversely, if a lower semicontinuous function is outer γ-convex for all $\gamma > 0$ then it must be convex. Indeed, if a function is outer γ-convex for all $\gamma > 0$ then it is convexlike (see [1]) and therefore, by lower semicontinuouity, this function is convex.

In [4], a weaker notion of generalized convexity, namely γ-convexlikeness was introduced. We recall that a function f is *γ-convexlike* if for all x_0, x_1 in D, satisfying $\|x_0 - x_1\| > \gamma$, there exists $z \in]x_0, x_1[$ such that

$$f(z) \leq (1 - \lambda) f(x_0) + \lambda f(x_1), \tag{3.2}$$

where $\lambda := \|x_0 - z\| / \|x_0 - x_1\|$.

Then, outer γ-convexity and γ-convexlikeness are equivalent for lower semicontinuous functions.

Proposition 3.1 ([5]). *Let f be lower semicontinuous. Then, f is outer γ-convex iff it is γ-convexlike.*

It is easy to see that a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ of order 4 is not convex on $\mathbb{R}^1$ iff $0 < 3b^2 - 8ac$. But f is outer γ-convex for a suitable γ as the following shows.

Corollary 3.2. *Suppose that a polynomial* $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ *of order 4 is not convex on* $\mathbb{R}^1$*. Then* f *is outer* γ*-convex iff* $a > 0$ *and* $\gamma \geq \frac{1}{2a}\sqrt{\frac{3(3b^2-8ac)}{2}}$.

Proof. Proposition 3.1 allows us to conclude that outer γ-convexity of a polynomial is equivalent to γ-convexlikeness. Therefore, f is outer γ-convex iff for all $x_0, x_1 \in \mathbb{R}^1$ and $x_1 - x_0 > \gamma$, there exists $\lambda \in]0,1[$ such that

$$f\left(x_0 + \lambda(x_1 - x_0)\right) \leq (1-\lambda)f(x_0) + \lambda f(x_1).$$

This inequality is equivalent to

$$g(x_1) = 6ax_1^2 - \left(4a(2-\lambda)p - 3b\right)x_1 + a(3 - 3\lambda + \lambda^2)p^2 - b(2-\lambda)p + c \geq 0,$$

where $p := x_1 - x_0$. Fix p and λ. Then, the polynomial $g(x_1)$ of order 2 is greater than 0 for all $x_1 \in \mathbb{R}^1$ iff $a > 0$ and

$$8a^2(1 - \lambda + \lambda^2)(x_1 - x_0)^2 \geq 9b^2 - 24ac \tag{3.3}$$

holds true for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$.

Now suppose that f is outer γ-convex. It follows from the above that $a > 0$ and (3.3) holds for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$. Since $\lambda \in [0,1]$, $0 < 1 - \lambda + \lambda^2 \leq 1$. Hence, by (3.2), $9b^2 - 24ac \leq 8a^2(x_1 - x_0)^2$ for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$. It follows that $0 < 3(3b^2 - 8ac) \leq 8a^2\gamma^2$.

Conversely, suppose that $a > 0$ and $0 < 3(3b^2 - 8ac) \leq 8a^2\gamma^2$. We prove that f is outer γ-convex. Assume the contrary that f is not outer γ-convex. Then, by (3.3), there exist $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$ and

$$8a^2(1 - \lambda + \lambda^2)(x_1 - x_0)^2 < 9b^2 - 24ac$$

for all $\lambda \in]0,1[$. It implies that $(x_1 - x_0)^2 \leq \gamma^2$, a contradiction. $\square$

It is well known that f is convex iff the Jenssen inequality holds, namely $x_1, \ldots, x_m \in D$ imply that $f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$ for all $\lambda_i \geq 0$, $i = 1, \ldots, m$ satisfying $\sum_{i=1}^m \lambda_i = 1$ (see, e.g. [8]).

Proposition 3.3. *If f is outer γ-convex then $x_1, \ldots, x_m \in D$ and*

$$\inf_{x \in \mathit{conv}\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m\}} \|x_i - x\| > \gamma \quad \text{for all } i = 1, \ldots, m$$

and $m \geq 2$ imply that there exist $\lambda_i > 0$, $i = 1, \ldots, m$, $\sum_{i=1}^m \lambda_i = 1$, $f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$. If additionally f is lower semicontinuous, the converse is true.

Proof. Suppose that f is outer γ-convex. We apply the argument given in the proof of Proposition 2.5 again, with "$\sum \lambda_i x_i \in M$" replaced by "$f(\sum \lambda_i x_i) \leq \sum \lambda_i f(x_i)$", to obtain the desired result.

Conversely, since the above condition holds true for $m = 2$, f is γ-convexlike. Hence, by Proposition 3.1, f is outer γ-convex. $\square$

Note that the sufficiency of Proposition 3.3 fails to be true without the assumption on the lower semi continuity of f.

A property of generalized convex functions is called an infection property if this property transmits to other places after once appearing somewhere. Phu and Hai ([6]) showed that γ-convex functions on $\mathbb{R}$ possess some infection properties. Outer γ-convex functions also possess an infection property as the following proposition shows.

Proposition 3.4. *Let $f : D \subset X \to \mathbb{R}$ be outer γ-convex and $x_0 \in D$ satisfy $\liminf_{x \to x_0} f(x) = -\infty$. If there exists $y \in D$ satisfying*

$$\|y - x_0\| \geq 2\gamma \tag{3.4}$$

then there is some

$$z \in \left[x_0 + \gamma \frac{y - x_0}{\|y - x_0\|}, x_0 + 2\gamma \frac{y - x_0}{\|y - x_0\|}\right]$$

such that $\liminf_{x \to z} f(x) = -\infty$.

Proof. Assume that $x_0 = \lim_{m \to +\infty} x_m$ and $\lim_{m \to +\infty} f(x_m) = -\infty$ with some $\{x_m\} \subset D$. Since $\|y - x_0\| \geq 2\gamma$, we also assume that $\|y - x_m\| > \gamma$ for all m. Set $s_m := (y - x_m)/\|y - x_m\|$. Because f is outer γ-convex, there exist $z_m^j = (1 - \lambda_m^j)x_m + \lambda_m^j y, j = 1, 2$ satisfying

$$\|z_m^2 - z_m^1\| \leq \gamma, \quad x_m + \gamma s_m \in [z_m^1, z_m^2], \tag{3.5}$$

and

$$f(z_m^j) \leq (1 - \lambda_m^j) f(x_m) + \lambda_m^j f(y), \tag{3.6}$$

where $\lambda_m^j := \|x_m - z_m^j\|/\|y - x_m\|$. Since $\{\lambda_m^j\} \subset [0, 1]$, we can assume that $\lambda_m^j \to \lambda^j \in [0, 1]$ as $m \to +\infty$. It follows that $z_m^j \to z^j := (1 - \lambda^j)x_0 + \lambda^j y$ as $m \to +\infty$, $j = 1, 2$. We now consider the following cases:

a) If $\lambda^2 \neq 1$, i.e., $z_m^2 \not\to y$ as $m \to +\infty$. This together with (3.6) yields

$$\begin{aligned} \liminf_{m \to +\infty} f(z_m^2) &\leq \liminf_{m \to +\infty} \{(1 - \lambda_m^2) f(x_m) + \lambda_m^2 f(y)\} \\ &\leq \liminf_{m \to +\infty} (1 - \lambda_m^2) f(x_m) + \limsup_{m \to +\infty} \lambda_m^2 f(y) \\ &= -\infty. \end{aligned}$$

Therefore

$$\liminf_{m\to+\infty} f(z_m^2) = -\infty.$$

That is,

$$\liminf_{x\to z^2} f(x) = -\infty.$$

Since $z_m^2 \in [x_m+\gamma s_m, x_m+2\gamma s_m]$, we conclude that $z^2 \in [x_0+\gamma s_0, x_0+2\gamma s_0]$.
b) If $\lambda^2 = 1$, i.e., $z_m^2 \to y$ as $m \to +\infty$. Then, by (3.5), we conclude that $\|x_0 - y\| = 2\gamma$, $z_m^1 \to z^1 = x_0 + \gamma s_0$ as $m \to +\infty$ and therefore $\lambda^1 \neq 1$. Applying the argument given in case **a**) again, with "z_m^2" replaced by "z_m^1", we get

$$\liminf_{x\to z^1} f(x) = -\infty.$$

This completes our proof. □

Note that the number 2γ in (3.4) is best possible. This is illustrated by

$$f(x) \colon = \begin{cases} 0 & \text{if } x \in \{0\} \cup [a,b] \\ \frac{1}{x(x-a)} & \text{if } x \in]0,a[\end{cases}$$

($1 < b < 2$ and $b - 1 < a < 1$). Obviously, f is outer γ-convex on $D \colon = [0,b]$ with $\gamma \colon = 1$. Choose $x_0 \colon = 0$ and $y \colon = b$ then $\liminf_{x\to x_0} f(x) = -\infty$ and $y - x_0 = b < 2\gamma$. In this case, $\lim_{x\to z} f(x) = 0$ for all $z \in [x_0+\gamma, y]$ and the conclusion of Proposition 3.4 is false.

In the next section, a Lipschitz condition is assumed and therefore, the infection property above does not occur.

4. The Outer γ-Convexity of Functions and their Epigraphs

Similar to convex functions, outer γ-convex functions can be characterized by their epigraphs.

Theorem 4.1. *Suppose that* $\|(x,t)\|_1 : = \max\{\|x\|, |t|\}$ *for all* $x \in X, t \in \mathbb{R}$. *If* $\operatorname{epi} f$ *is outer* γ*-convex then* f *is outer* γ*-convex. Conversely, if an outer* γ*-convex* f *is Lipschitz continuous with constant* $\alpha > 1$ *(*$\alpha \in [0,1]$*, respectively) then* $\operatorname{epi} f$ *is outer* $\alpha\gamma$*-convex (outer* γ*-convex, respectively).*

Proof. Suppose that $\operatorname{epi} f$ is outer γ-convex and $x_0, x_1 \in D$ such that $\|x_1 - x_0\| > \gamma$. Then

$$\|\,(x_1, f(x_1)) - (x_0, f(x_0))\,\|_1 \geq \|x_1 - x_0\| > \gamma.$$

It follows that there exist

$$\begin{aligned} A_0 &:= (x_0, f(x_0))\,, \\ A_1, \dots, A_k &:= (x_1, f(x_1)) \in [(x_0, f(x_0))\,, (x_1, f(x_1))] \cap \operatorname{epi} f \end{aligned}$$

such that

$$\|A_{i+1} - A_i\|_1 \leq \gamma \quad \text{with} \quad i = 0, 1, \dots, k-1.$$

Suppose that $A_i = (z_i, t_i)$. Then

$$\|z_{i+1} - z_i\| \leq \|A_{i+1} - A_i\|_1 \leq \gamma \quad \text{with} \quad i = 0, 1, \dots, k-1.$$

On the other hand, since

$$A_i = (z_i, t_i) \in [(x_0, f(x_0)), (x_1, f(x_1))] \cap \operatorname{epi} f,$$

we get

$$f(z_i) \leq t_i = (1 - \lambda_i) f(x_0) + \lambda_i f(x_1)$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|, \quad i = 1, 2, \dots, k-1$. That is, f is outer γ-convex.

Conversely, if an outer γ-convex function f is Lipschitz continuous with constant $\alpha > 1$ ($\alpha \in [0,1]$, respectively) then $\operatorname{epi} f$ is outer $\alpha\gamma$-convex (outer γ-convex, respectively). Indeed, let

$$Y_0 = (x_0, t_0), Y_1 = (x_1, t_1) \in \operatorname{epi} f.$$

Obviously, f is continuous on $[x_0, x_1]$.

Hence, $\{(x,t) \in \operatorname{epi} f : x \in [x_0, x_1]\}$ is closed. Assume without loss of generality, that

$$Y_0 = (x_0, f(x_0)), Y_1 = (x_1, f(x_1)).$$

Suppose

$$\|Y_1 - Y_0\|_1 > \alpha\gamma \quad \text{with} \quad \alpha > 1$$

($\|Y_1 - Y_0\|_1 > \gamma$ with $0 \leq \alpha \leq 1$, respectively). Then, by $\|(x,t)\|_1 := \max\{\|x\|, |t|\}$,

$$\alpha \|x_1 - x_0\| \geq |f(x_1) - f(x_0)|$$

implies

$$\|x_1 - x_0\| \geq \frac{\|Y_1 - Y_0\|_1}{\alpha} > \gamma \quad \text{with} \quad \alpha > 1$$

($\|x_1 - x_0\| = \|Y_1 - Y_0\|_1 > \gamma$ with $0 \le \alpha \le 1$, respectively). By the outer γ-convexity of f, there exist $z_0 := x_0, z_1, \dots, z_k := x_1 \in [x_0, x_1]$ satisfying (2.1) and (3.1). Set

$$A_i := (z_i, (1-\lambda_i) f(x_0) + \lambda_i f(x_1)),$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|, \;\; i = 0, 1, \dots, k$. It follows that

$$A_0, A_1, \dots, A_k \in [Y_0, Y_1] \cap \operatorname{epi} f$$

and

$$\|A_{i+1} - A_i\|_1 \le \alpha\gamma, \; i = 0, 1, \dots, k-1 \quad \text{with} \quad \alpha > 1$$

($\|A_{i+1} - A_i\|_1 \le \gamma$, $i = 0, 1, \dots, k-1$ with $0 \le \alpha \le 1$, respectively). Hence, $\operatorname{epi} f$ is outer $\alpha\gamma$-convex with $\alpha > 1$ ($\operatorname{epi} f$ is outer γ-convex with $0 \le \alpha \le 1$, respectively), and the proof is complete. $\square$

Corollary 4.2. *Suppose that* $\|(x,t)\|_1 := \max\{\|x\|, |t|\}$ *for all* $x \in X$, $t \in \mathbb{R}$ *and* f *is Lipschitz continuous with constant* $\alpha \in [0,1]$. *Then,* f *is outer* γ*-convex iff* $\operatorname{epi} f$ *is outer* γ*-convex.*

Note that the assumptions of norm and Lipschitz condition in Theorem 4.1 and Corollary 4.2 are really needed.

5. Concluding Remarks

Some sufficient conditions for some kinds of outer γ-convex functions, namely strictly γ-convex functions and γ-convex functions, were given in [3] and [6]. Some sufficient conditions for outer γ-convex function will be a subject of another paper.

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