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THE ROLE OF AN INTEGRAL INEQUALITY IN THE STUDY OF CERTAIN DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we present an integral inequality and show how it can be used to study certain differential equations. Namely, we will see how to establish (global) existence results and determine the decay rates of solutions to abstract semilinear problems, reaction diffusion systems with time dependent coefficients and fractional differential problems. A nonlinear singular version of the Gronwall inequality is also presented.

Key words and phrases: Abstract semilinear Cauchy problem, analytic semigroup, decay rate, fractional derivative, fractional operator, global existence, singular kernel.

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1. INTRODUCTION AND PRELIMINARIES

Our purpose here is to survey the recent works of the present author together with some of his collaborators on the role of an integral inequality in developing certain results in the prior literature.

In this section we present the integral inequality in question together with its proof from [21]. Then, we prepare some material which will be needed later. Since we will be dealing with different results and applications published in different papers, it will also be our task here in this section to unify the notation.

We denote by $X := L^p(\Omega)$, $p > 1$ and $W^{m,p}(\Omega)$, $p > 1$, $m \geq 1$, where Ω is a bounded domain in $\mathbb{R}^n$, the usual Lebesgue space and Sobolev space, respectively. The space $C^\nu\left(\bar{\Omega}\right)$, $\nu \geq 0$, is the Banach space of $[\nu]$-times continuously differentiable functions in $\bar{\Omega}$ whose $[\nu]$-th order derivatives are Hölder continuous with exponent $\nu - [\nu]$, so that $C^0\left(\bar{\Omega}\right) = C\left(\bar{\Omega}\right)$ and

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$C^1\left(\bar{\Omega}\right)$ are the Banach spaces of continuous and continuously differentiable functions in $\bar{\Omega}$, respectively.

We designate by $-A$ a sectorial operator (see [9]) with $\operatorname{Re}\sigma\left(A\right)>b>0$ where $\operatorname{Re}\sigma\left(A\right)$ denotes the real part of the spectrum of A. We may define the fractional operators A^α, $0\leq\alpha\leq 1$ in the usual way on $D\left(A^\alpha\right)=X^\alpha$. The space X^α endowed with the norm $\left\|x\right\|_\alpha=\left\|A^\alpha x\right\|$ is a Banach space. The operator $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in X.

Our key inequality in this paper is the following (see [21]).

Lemma 1.1. *If $\lambda,\nu,\omega>0$, then for any $t>0$ we have*

$$t^{1-\nu}\int_0^t(t-s)^{\nu-1}s^{\lambda-1}e^{-\omega s}ds\leq C,$$

where C is a positive constant independent of t. In fact,

$$C=\max\left\{1,2^{1-\nu}\right\}\Gamma(\lambda)(1+\lambda/\nu)\omega^{-\lambda}.$$

Proof. Let $I(t)$ denote the left-hand side of the relation in the lemma. By a change of variables we find

$$I(t)=t^\lambda\int_0^1(1-\xi)^{\nu-1}\xi^{\lambda-1}e^{-\omega t\xi}d\xi.$$

Notice that,

$$t^\lambda(1-\xi)^{\nu-1}\xi^{\lambda-1}e^{-\omega t\xi}\leq\begin{cases}\max(1,2^{1-\nu})t^\lambda\xi^{\lambda-1}e^{-\omega t\xi}, & 0\leq\xi\leq\frac{1}{2}\\ \\ 2(1-\xi)^{\nu-1}\Gamma(\lambda+1)\omega^{-\lambda}, & \frac{1}{2}<\xi\leq 1.\end{cases}$$

Therefore,

$$I(t)\leq\max(1,2^{1-\nu})\Gamma(\lambda)(1+\lambda/\nu)\omega^{-\lambda}.$$

$\square$

We will also need the lemmas below (see [9] for the proofs)

Lemma 1.2. *If $0\leq\alpha\leq 1$, then $D\left(A^\alpha\right)\subset C^\nu\left(\bar{\Omega}\right)$ for $0\leq\nu<2\alpha-\frac{n}{p}$.*

Lemma 1.3. *If $0\leq\alpha\leq 1$, then $\left\|A^\alpha e^{-tA}\right\|_p\leq c_1t^{-\alpha}e^{-bt}$, $t>0$ for some positive constant c_1.*

Lemma 1.4. *Let $\alpha\in[0,1)$ and $\beta\in\mathbb{R}$. There exists a positive constant $C=C(\alpha,\beta)$ such that*

$$\int_0^t s^{-\alpha}e^{\beta s}ds\leq\begin{cases}Ce^{\beta t}, & \text{if }\beta>0\\ C(t+1), & \text{if }\beta=0\\ C, & \text{if }\beta<0.\end{cases}$$

Lemma 1.5. *Let $a(t)$, $b(t)$, $K(t)$, $\psi(t)$ be nonnegative, continuous functions on the interval $I=(0,T)$ $(0<T\leq\infty)$, $\Phi:(0,\infty)\to\mathbb{R}$ be a continuous, nonnegative and nondecreasing function, $\Phi(0)=0$, $\Phi(u)>0$ for $u>0$ and let $A(t)=\max_{0\leq s\leq t}a(s)$, $B(t)=\max_{0\leq s\leq t}b(s)$. Assume that*

$$\psi(t)\leq a(t)+b(t)\int_0^t K(s)\Phi(\psi(s))ds,\quad t\in I.$$

Then

$$\psi(t)\leq W^{-1}\left[W(A(t))+B(t)\int_0^t K(s)ds\right],\quad t\in(0,T_1),$$

where $W(v)=\int_{v_0}^v\frac{d\sigma}{\Phi(\sigma)}$, $v\geq v_0>0$, W^{-1} is the inverse of W and $T_1>0$ is such that $W(A(t))+B(t)\int_0^t K(s)ds\in D(W^{-1})$ for all $t\in(0,T_1)$.

This result may be found in [1] for instance.

We caution the reader that due to space considerations we are unable to discuss all the prior literature on the different problems presented in this paper. Our main objective is to emphasize and highlight the role played by the integral inequality (Lemma 1.1) in improving and extending previous results for a variety of problems.

2. ABSTRACT SEMILINEAR PROBLEMS

Let us consider the problem

$$\begin{cases} u_t + Au = f(t,u), \ u \in X \\ u(0) = u_0 \in X, \end{cases} \tag{2.1}$$

where A is a sectorial operator with $\operatorname{Re}\sigma(A) > b > 0$. The function $f(t,u)$ satisfies

$$\|f(t,u)\| \le t^{\kappa}\eta(t)\,\|A^{\alpha}u\|^{m}, \quad m > 1, \ \kappa \ge 0, \tag{2.2}$$

where $\eta(t)$ is a nonnegative continuous function. Solutions of the differential problem (2.1) coincide with solutions of the integral equation

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s,u(s))ds, \quad 0 < t \le T \tag{2.3}$$

with continuous $u : (0,T) \to X^{\alpha}$ and $f : t \longmapsto f(t,u(t))$.

In [19], Medved' considered this problem and proved a global existence result. He also proved that $\lim_{t\to\infty}\|u(t)\|_{\alpha} = 0$ provided that

$$t \longmapsto t^{rq\alpha}\int_0^t \eta(s)^{rq}e^{rq[(1-m)b+m\varepsilon]s}ds \tag{2.4}$$

is bounded on $(0,\infty)$ for some positive real numbers ε, q and r. This has been established for a certain range of values for α. In fact, the decay rate there was found to be exponential. The idea was to take the α-norm $\|\cdot\|_{\alpha}$ of both sides of the equation (2.3) and use the hypothesis (2.2) and Lemma 1.3 to obtain

$$\Psi(t) \le d\,\|u_0\| + dt^{\alpha}\int_0^t (t-s)^{-\alpha}e^{b(1-m)s}s^{\kappa-m\alpha}\eta(s)\Psi(s)^{m}ds$$

for a certain function $\Psi(t)$. Medved' then appealed to a nonlinear singular version of the Gronwall inequality which he proved earlier in [18]. This latter result gives bounds for solutions of inequalities of the type

$$\psi(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi^{m}(s)ds, \quad \beta > 0, \ \gamma > 0 \tag{2.5}$$

where $m > 1$ (the linear case ($m = 1$) can be found, for instance, in [9]). Medved' used the decomposition

$$\begin{aligned} &\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\Psi(s)^{m}ds \\ &\qquad \le \left(\int_0^t (t-s)^{2(\beta-1)}e^{2\varepsilon s}ds\right)^{\frac{1}{2}}\left(\int_0^t s^{2(\gamma-1)}F(s)^2e^{-2\varepsilon s}\Psi(s)^{2m}ds\right)^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

and Lemma 1.5. In [13], Kirane and Tatar improved considerably the latter and the former results by using the above inequality in Lemma 1.1 and the decomposition

$$\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\Psi(s)^m ds \le \left(\int_0^t (t-s)^{2(\beta-1)}s^{2(\gamma-1)}e^{-2s}ds\right)^{\frac{1}{2}}\left(\int_0^t F(s)^2e^{2s}\Psi(s)^{2m}ds\right)^{\frac{1}{2}} \tag{2.7}$$

instead of the decomposition (2.6). The assumption (2.4) has been relaxed and the range of values of α has been enlarged. In fact, the gap which was in [19] was filled. We established an exponential decay and a power type decay for those values of α which were not considered in [19]. The estimates are proved in the space $D(A^\alpha)$, then using the Lemma 1.2 we pass to the space $C^\mu(\bar{\Omega})$, $0<\mu<2\alpha-\frac{n}{p}$.

Then, in the same paper [13], these results were extended to the case of abstract semilinear *functional* differential problems of the form

$$\begin{cases} \frac{du}{dt}+Au=f(t,u(t+\theta)),\ u\in X,\ \theta\in[-r,0] \\ u(0)=u_0\in X \end{cases}$$

and *integro-differential* problems of the form

$$\begin{cases} \frac{du}{dt}=\int_0^t k(t-s)Au(s)ds+f(t,u),\ u\in X \\ u(0)=u_0\in X. \end{cases}$$

3. SOME FURTHER EXTENSIONS

The results stated in the previous section were extended to other differential problems with different nonlinearities. In [26], the present author considered the following abstract problem

$$\begin{cases} \frac{du}{dt}+Au=F\left(t,u(t),\int_0^t l(t,s)f(s,u(s)ds\right),\ t\in I=[0,T] \\ u(0)=u_0\in X, \end{cases}$$

where $f:I\times X\to X$ and $F:I\times X\times X\to X$ satisfy

(H1) There exist continuous functions $\varphi:I\to[0,\infty)$ and $q:I\to[0,\infty)$ such that

$$\|f(t,u)\|\le\varphi(t)\theta(\|u\|),\quad u\in X,\ t\in I$$

for some continuous nondecreasing function $\theta:[0,\infty)\to[0,\infty)$ satisfying

$$\theta(\sigma(t))^2\le q(t)\theta(\sigma(t)^2).$$

(H2) There exists a continuous function $\psi:I\to[0,\infty)$ such that

$$\|F(t,u,v)\|\le\psi(t)\left(\|u\|+\|v\|\right),\quad u,v\in X,\ t\in I.$$

After proving quite a general well-posedness result, we established an exponential decay result for singular kernels of the form

$$l(t,s)=l(t-s)=(t-s)^{-\beta}e^{-\gamma(t-s)},\quad \beta\in(0,1),\ \gamma>0$$

and for θ of polynomial type $\theta(r):=r^m$. Observe here that the nonlinearity we are dealing with is somewhat different from the previous one. If we take $X=L^p(\Omega)$, $p>1$, then it is the L^p-norm we are considering here instead of the $\alpha-$norm, that is,

$$\|f(t,u)\|_p\le t^\mu\chi(t)\|u\|_p^m,\quad \mu\ge0.$$

This improves several results in the prior literature for $m = 1$ and time independent (or bounded $\varphi(t)$) nonlinearities.

Then, we can cite the work in [17] dealing with the integro-differential problem

$$\begin{cases} \frac{du}{dt} + Au = f(t, u(t)) + \int_0^t g\left(t, s, u(s), \int_0^s K(s, \tau, u(\tau)\right) d\tau, \quad t \in I = [0, T] \\ u(0) = u_0 \in X \end{cases}$$

and where again an exponential decay result was proved using the integral inequality in Lemma 1.1. The global existence is proved, in a more general setting in [16] for a problem with non-local conditions of the form

$$u(0) + h(t_1, ..., t_p, u) = u_0$$

and with delays in the arguments of the solution u. Namely, the problem treated there was

$$\begin{cases} \frac{du}{dt} + Au \\ = F\left(t, u\left(\sigma_1(t)\right), \int_0^t g\left(t, s, u\left(\sigma_2(s)\right), \int_0^s K\left(s, \tau, u\left(\sigma_3(\tau)\right)\right) d\tau\right) ds\right) \\ u(0) + h(t_1, ..., t_p, u(\cdot)) = u_0 \in X. \end{cases}$$

4. THE HEAT EQUATION

In this part of the paper we consider the following integral inequality

$$\varphi(t, x) \le k(t, x) + l(t, x) \int_\Omega \int_0^t \frac{F(s)\varphi^m(s, y)}{(t-s)^{1-\beta} |x-y|^{n-\alpha}} dyds, \quad x \in \Omega,\ t > 0 \tag{4.1}$$

where Ω is a domain in $\mathbb{R}^n$ ($n \ge 1$) (bounded or possibly equal to $\mathbb{R}^n$), the functions $k(t, x)$, $l(t, x)$ and $F(t)$ are given positive continuous functions in t. The constants $0 < \alpha < n$, $0 < \beta < 1$ and $m > 1$ will be precised below.

The interest in this inequality which is singular in both time and space is motivated by the semilinear parabolic problem (in case $\Omega = \mathbb{R}^n$)

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + u^m(t, x),\ x \in \mathbb{R}^n,\ t > 0,\ m > 1 \\ u(0, x) = u_0(x),\ x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

This problem (and also on a bounded domain) has been extensively studied by many researchers, see for instance the survey paper by Levine [15]. Several results on global existence, blow up in finite time and asymptotic behavior have been found. These results depend in general on the dimension of the space n, the exponent m and the initial data $u_0(x)$. In particular, global existence has been proved for sufficiently small initial data (together with an assumption on n and m). Using the fundamental solution $G(t, x)$ of the heat equation we can write this problem in the integral form

$$u(t, x) = \int_{\mathbb{R}^n} G(t, x-y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} G(t-s, x-y) u^m(s, y) dyds. \tag{4.3}$$

Recalling the Solonnikov estimates [25]

$$|G(t-s, x-y)| \le \frac{C}{\left(|t-s|^{1/2} + |x-y|\right)^n},$$

it is clear that we can end up with a particular form of the inequality (4.1).

Notice here that the integral inequality (4.1) is not merely an extension of the singular nonlinear Gronwall inequality (2.5) discussed above to the case of two variables. This case has been treated by Medved' in [20]. Namely, the author considered an inequality of the form

$$u(x,y) \leq a(x,y) + \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} F(s,t)\omega(u(s,t))dsdt,$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$e^{-qt}[\omega(u)]^q \leq R(t)\omega(e^{-qt}u^q)$$

for some $q > 0$ and $R(t)$ a continuous nonnegative function. His results, in turn, may be improved by applying a similar decomposition to (2.7) twice.

The inequality (4.1) is different and the technique previously mentioned is not applicable in this situation. In [27], we have been forced to combine this technique with the Hardy-Littlewood-Sobolev inequality.

Lemma 4.1 (see [11, p. 117])**.** *Let $u \in L^p(\mathbb{R}^n)$ $(p > 1)$, $0 < \gamma < n$ and $\frac{\gamma}{n} > 1 - \frac{1}{p}$, then $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{\gamma}{n} + \frac{1}{p} - 1$. Also the mapping from $u \in L^p(\mathbb{R}^n)$ into $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^n)$ is continuous.*

We found sufficient conditions involving some L^p-norms of l and k with F yielding existence and estimations of solutions on some intervals.

Theorem 4.2. *Assume that the constants α, β and m are such that $0 < \alpha < \beta n$, $0 < \beta < 1$ and $m > 1$.*

(i) *If $\Omega = \mathbb{R}^n$, then for any r satisfying $\max\left(\frac{(m-1)n}{\alpha}, \frac{m}{\beta}\right) < r < \frac{mn}{\alpha}$, we have*

$$\|\varphi(t,x)\|_r \leq U_{p,r,\rho}(t)$$

with

$$U_{p,r,\rho}(t) = 2^{\frac{m(p-1)}{r}} K(t)^{\frac{1}{p}} \times \left[1 - 2^{m(p-1)}(m-1)C_1^{p-1}C_2^p K(t)^{m-1} L(t) e^{\varepsilon pt} \int_0^t e^{-\varepsilon ps} F^p(s)ds\right]^{\frac{m}{(1-m)r}},$$

where $K(t) = \max_{0\leq s\leq t} \|k(s,\cdot)\|_r^p$, $L(t) = \max_{0\leq s\leq t} \|l(s,\cdot)\|_\rho^p$, $p = r/m$ and $\rho = \frac{nr}{\alpha r-(m-1)n}$ for some $\varepsilon > 0$. Here C_1 and C_2 are the best constants in Lemma 1.4 and Lemma 4.1, respectively. The estimation is valid as long as

$$K(t)^{m-1}L(t)e^{\varepsilon pt}\int_0^t e^{-\varepsilon ps}F^p(s)ds \leq 1/2^{m(p-1)}(m-1)C_1^{p-1}C_2^p. \tag{4.4}$$

(ii) *If Ω is bounded, then*

$$\|\varphi(t,x)\|_{\tilde{r}} \leq U_{p,r,\rho}(t)$$

for any $\tilde{r} \leq r$ where p, r and ρ are as in (i). If moreover, $r < n/(n\beta - \alpha)$ (but not necessarily $r > (m-1)n/\alpha$), that is, $\frac{m}{\beta} < r < \min\left(\frac{mn}{\alpha}, \frac{n}{n\beta-\alpha}\right)$, then this estimation holds for any $\frac{1}{\beta} < p \leq \frac{r}{m}$ provided that $\rho > \frac{nr}{n-(n\beta-\alpha)r}$.

From (4.4) it can be seen that the growth of $K(t)$ may be "controlled" by $L(t)$ and $F(t)$. That is, if $K(t)$ is large then we can assume $L(t)$ and/or $F(t)$ small enough to get existence on an arbitrarily large interval of time. In fact, for the case of the semilinear parabolic (heat) problem (4.2), it is known that

$$\int_{\mathbb{R}^n} G(t,x-y)u_0(y)dy \leq u_0^M(x)$$

where $u_0^M(x)$ is the *maximal* function defined by

$$u_0^M(x) = \sup \frac{1}{|R|} \int_R |u_0(y)|\, dy.$$

The $\sup$ is taken over all cubes R centered at x and having their edges parallel to the coordinate axes. Moreover, the L^p-norm of u_0^M is less than a constant times the L^p-norm of u_0. This means that if $u_0 \in L^p(\mathbb{R}^n)$, we will be left with a condition involving $u_0^M(x)$ only (see (4.3)).

Moreover, it is proved in [27] that

Corollary 4.3. *Suppose that the hypotheses of Theorem 4.2 hold. Assume further that $k(t,x)$ and $l(t,x)$ decay exponentially in time, that is $k(t,x) \le e^{-\tilde{k}t}\bar{k}(x)$ and $l(t,x) \le e^{-\tilde{l}t}\bar{l}(x)$ for some positive constants $\tilde{k}$ and $\tilde{l}$. Then $\varphi(t,x)$ is also exponentially decaying to zero i.e.,*

$$\|\varphi(t,x)\|_v \le C_3 e^{-\mu t}, \quad t > 0$$

for some positive constants C_3 and μ provided that

$$\left\|\bar{k}(x)\right\|_r^{m-1} \left\|\bar{l}(x)\right\|_\rho \int_0^\infty F^p(s)ds \le \frac{1}{2^{m(p-1)}}(m-1)C_4^{p-1}C_2^p,$$

where C_4 is the best constant in Lemma 1.4 and the other constants are as in (i) and (ii) of Theorem 4.2.

Finally, for the nonlinear singular inequality

$$\varphi(t,x) \le k(t,x) + l(t,x) \int_\Omega \int_0^t \frac{s^\delta F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}|x-y|^{n-\alpha}} dyds, \quad x \in \Omega,\ t > 0$$

we can prove an interesting result yielding power type decay without imposing a power type decay for $l(t,x)$.

Corollary 4.4. *Suppose that the hypotheses of Theorem 4.2 hold. Assume further that $k(t,x) \le t^{-\hat{k}}\bar{k}(x)$ and $1 + \delta p' - mp' \min\{\hat{k}, 1-\beta\} > 0$. Then any $\varphi(t,x)$ satisfying the above inequality is also polynomially decaying to zero*

$$\|\varphi(t,x)\|_v \le C_5 t^{-\omega}, \quad C_5, \omega > 0$$

provided that

$$\left\|\bar{k}(x)\right\|_r^{m-1} L(t) \int_0^t e^{\varepsilon ps} F^p(s)ds \le \frac{1}{2^{m(p-1)}}(m-1)C_6^{p-1}C_2^p$$

where C_6 is the best constant in Lemma 1.1.

5. REACTION DIFFUSION SYSTEMS

In this section we are interested in systems of reaction-diffusion equations of the form

$$\begin{cases} u_t = d_1\Delta u - r_1(t)f_1(u)w^\gamma - r_2(t)f_2(u)z^\eta, & x \in \Omega,\ t > 0 \\ w_t = d_2\Delta w + r_1(t)f_1(u)w^\gamma + r_2(t)f_2(u)z^\eta - aw, & x \in \Omega,\ t > 0 \\ v_t = d_1\Delta v - r_3(t)f_3(v)w^\sigma - r_4(t)f_4(v)z^\rho, & x \in \Omega,\ t > 0 \\ z_t = d_1\Delta z + r_3(t)f_3(v)w^\sigma + r_4(t)f_4(v)z^\rho - az, & x \in \Omega,\ t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0 \\ (u,w,v,z)(x,0) = (u_0,w_0,v_0,z_0)(x), & x \in \Omega \end{cases}$$

where Ω is a bounded region in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the diffusion coefficients d_i, $i = 1,2,3,4$ and a are positive constants and the exponents $\gamma, \eta, \sigma, \rho$ are greater than one. It is also assumed that

(i) $\|u_0\|_1, \|w_0\|_1, \|v_0\|_1, \|z_0\|_1 > 0$;
(ii) f_i, $i = 1,2,3,4$ are nonnegative C^1-functions on $[0,\infty)$;
(iii) $f_i(0) = 0$, and $f_i(y) > 0$ if and only if $y > 0$, $i = 1,2,3,4$;
(iv) $1 < \eta \le \rho$ and $1 < \sigma \le \gamma$.

There are very few papers dealing with systems involving time-dependent nonlinearities and probably the only paper which treated the question of asymptotic behavior for reaction diffusion systems is the one by Kahane [12]. The author considered the system

$$\begin{cases} -u_t + Lu = f(x,t,u,v), & \text{in } \Omega \times (0,\infty) \\ -v_t + Mv = g(x,t,u,v), & \text{in } \Omega \times (0,\infty) \end{cases}$$

with boundary conditions of Robin type and where L and M are uniformly elliptic operators. He proved that the solution converges to the stationary state provided that

$$f(x,t,u,v) \to \bar{f}(x,u,v)$$

and

$$g(x,t,u,v) \to \bar{g}(x,u,v)$$

uniformly in Ω and (u,v) in any bounded subset of the first quadrant in $\mathbb{R}^2$ and the matrix formed by the partial derivatives $\bar{f}_u, \bar{f}_v, \bar{g}_u$ and $\bar{g}_v$ satisfies a column diagonal dominance type condition. This cannot be applied in our present case as we are going to consider unbounded coefficients

$$r_i(t) := t^{k_i} g_i(t), \quad k_i \ge 0,\ i = 1,2,3,4$$

where $g_i(t)$ are continuous and square integrable on $(0,\infty)$.

By standard arguments it can be seen that the operators in the system are sectorial and that they generate analytic semigroups in $L^p(\Omega)$. Then, these semigroups are shown to be *exponentially stable* in the sense of Lemma 1.3. Also, using the existing methods (fixed points theorems, a priori boundedness, maximum principle, Lyapunov functionals), one can easily show that for nonnegative continuous (on $\bar{\Omega}$) initial data there exists a unique nonnegative global solution bounded pointwise by a certain positive constant (equal to $\|u_0\|_\infty$ and $\|v_0\|_\infty$ in case of u and v, respectively). Making use of this and the fact that f_i are bounded, it is shown in [14] that solutions of the weak formulation

$$w(t) = e^{-tB_p} w_0 + \int_0^t e^{-(t-\tau)B_p} \left\{ r_1(\tau) f_1(u) w^\gamma + r_2(\tau) f_2(u) z^\eta \right\} d\tau$$

and

$$z(t) = e^{-tG_p} z_0 + \int_0^t e^{-(t-\tau)G_p} \left\{ r_3(\tau) f_3(v) w^\sigma + r_4(\tau) f_4(v) z^\rho \right\} d\tau$$

where B_p and G_p defined by

$$\begin{aligned} D(B_p) = D(G_p) &:= \left\{ y \in W^{2,p}(\Omega) : \frac{\partial y}{\partial \nu} |_{\partial\Omega} = 0 \right\} \\ B_p y &:= -(d_2 \Delta - a) y \\ G_p y &:= -(d_4 \Delta - a) y \end{aligned}$$

are exponentially decaying to 0. Then, we prove that the components u and v converge exponentially to u_∞ and v_∞ (the equilibrium state), respectively. Here again, our integral inequality in Lemma 1.1 plays an important role in the proof.

6. A CONVECTION PROBLEM

Another problem where we can see the efficiency of the integral inequality of Lemma 1.1 is the following initial value problem which appears in thermal convection flow

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = \Delta v - \tau g + h - \nabla\pi, & x\in\Omega,\ t>0, \\ \nabla\cdot v = 0, & x\in\Omega,\ t>0, \end{cases} \tag{6.1}$$

$$\partial_t\tau + (v\cdot\nabla)\tau = \Delta\tau, \quad x\in\Omega,\ t>0,$$

$$v(x,t)=0,\ \tau(x,t)=\xi(x,t), \quad x\in\Gamma,\ t>0,$$

$$v(x,0)=v_0(x),\ \tau(x,0)=\tau_0(x), \quad x\in\Omega,$$

where Ω is a bounded region in $\mathbb{R}^N$ ($N\geq 2$) with smooth boundary Γ.

This problem has been studied by Hishida in [10]. A quite general well-posedness result has been established there. However, the global existence result and the exponential decay were proved only for sufficiently small initial data and for ϕ satisfying the condition

$$\|\nabla\phi\|_\infty = O(e^{-\omega t}) \text{ with } \omega>0$$

where the function $\phi=\phi(x,t)$ is solution of

$$\begin{cases} \partial_t\phi = \Delta\phi, & x\in\Omega,\ t>0, \\ \phi(x,t)=\xi(x,t), & x\in\Gamma,\ t>0, \\ \phi(x,0)=\phi_0(x), & x\in\Omega \end{cases}$$

and $\phi_0=\phi_0(x)$ is defined by

$$\begin{cases} \Delta\phi_0 = 0, & \text{in } \Omega \\ \phi_0(x)=\xi(x,0), & \text{on } \Gamma. \end{cases}$$

In [5], the present author with Furati and Kirane improved these results in at least two directions. First, the class of functions ϕ is enlarged to functions satisfying

$$\|\nabla\phi\|_\infty = O(e^{-\omega t}) \text{ with } \omega\geq 0$$

and further to functions ϕ such that

$$\|\nabla\phi\|_\infty = O(t^{-\omega}) \text{ with } \omega\geq 0.$$

Next, combining the Gronwall-Bihari inequality (Lemma 1.5) and the integral inequality (Lemma 1.1), we were able to consider *large* initial data. To this end one has to reduce problem (6.1) to an abstract Cauchy problem of the form

$$\begin{cases} \frac{dv}{dt} + A_p v = F(v,\theta), & t>0,\ v(0)=v_0 \\ \frac{d\theta}{dt} + B_q\theta = G(v,\theta), & t>0,\ \theta(0)=\theta_0 \end{cases}$$

with

$$\begin{cases} F(v,\theta) = -\mathbf{P}_p(v\cdot\nabla)v - \mathbf{P}_p\theta g, \\ G(v,\theta) = -(v\cdot\nabla)v - (v\cdot\nabla)\phi. \end{cases}$$

Here $\mathbf{P}_p$ is the projection from $L^p(\Omega)^N$ onto $L^p_\sigma(\Omega)$ = the completion of $C^\infty_{0,\sigma}(\Omega) = \{\varphi\in C^\infty_0(\Omega)^N,\ \nabla\cdot\varphi=0\}$ in $L^p(\Omega)^N$, $1<p<\infty$ via the Helmholz decomposition $L^p(\Omega)^N = L^p_\sigma(\Omega)\oplus G_p(\Omega)$ with $G_p(\Omega)=\{\nabla\pi,\ \pi\in W^{1,p}(\Omega)\}$. The operators B_q and A_p are defined by

$$B_q = -\Delta \text{ with domain } D(B_q) = W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)$$

and

$$A_p = -\mathbf{P}_p\Delta \text{ with domain } D(A_p) = D(B_q)^N \cap L^p_\sigma(\Omega).$$

$-B_q$ and $-A_p$ generate then bounded analytic semigroups $\{\exp(-tB_q),\, t \geq 0\}$ on $L^q(\Omega)$ and $\{\exp(-tA_p),\, t \geq 0\}$ on $L^p_\sigma(\Omega)$ respectively. These semigroups are exponentially stable, that is

Lemma 6.1. *For each $\lambda_1 \in (0, \Lambda_1)$, $\alpha \geq 0$ and $\beta \geq 0$, we have*

$$\left\|A^\alpha e^{-tA} v\right\|_p \leq C_{\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t} \|v\|_p \text{ for } v \in L^p_\sigma(\Omega)$$

and

$$\left\|B^\beta e^{-tB} \theta\right\|_p \leq \hat{C}_{\beta,\lambda_1} t^{-\beta} e^{-\lambda_1 t} \|\theta\|_q \text{ for } \theta \in L^q(\Omega)$$

with some positive constants C_{α,λ_1} and $\hat{C}_{\beta,\lambda_1}$.

The problem can then be tackled via the formulation

$$\begin{cases} v(t) = e^{-tA_p} v_0 + \int_0^t e^{-(t-s)A_p} F(v,\theta)(s)ds, \\ \\ \theta(t) = e^{-tB_q}\theta_0 + \int_0^t e^{-(t-s)B_q} G(v,\theta)(s)ds. \end{cases}$$

The technique mentioned in Section 2 applies for these mild solutions and gives better results than the argument used in [10].

It is worth mentioning here that our argument works even for functions ϕ such that

$$\|\nabla\phi(t)\|_\infty = O(t^\tau), \quad \tau \geq 0$$

but with sufficiently small τ. We refer the reader to [5] for the details.

7. FRACTIONAL DIFFERENTIAL PROBLEMS

In this section we would like to present another type of differential problem where our integral inequality has proved to be very efficient. Let us consider the weighted Cauchy-type problem

$$\begin{cases} D^\alpha u(t) = f(t,u), \quad t > 0 \\ t^{1-\alpha} u(t)\,|_{t=0} = b, \end{cases} \tag{7.1}$$

where D^α is the fractional derivative (in the sense of Riemann-Liouville) of order $0 < \alpha < 1$ and $b \in \mathbb{R}$.

The function $f(t,u)$ satisfies the hypothesis:

(**F**) $f(t,u)$ is a continuous function on $\mathbb{R}^+ \times \mathbb{R}$ and is such that

$$|f(t,u)| \leq t^\mu \varphi(t)\, |u|^m, \qquad m > 1,\ \mu \geq 0,$$

where $\varphi(t)$ is a differentiable function on $\mathbb{R}^+$ with $\varphi(0) \neq 0$.

For the reader's convenience, we recall below the definition of the derivative of non-integer order.

Definition 7.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the integral exists.

Definition 7.2. The fractional derivative (in the sense of Riemann-Liouville) of order $0 < \alpha < 1$ of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as the left inverse of the fractional integral of f

$$D^\alpha f(t) = \frac{d}{dt}(I^{1-\alpha} f)(t).$$

That is

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds,$$

provided that the right side exists.

The reader is referred to [24] for more on fractional integrals and fractional derivatives.

For $h > 0$, we define the space

$$C_r^0([0,h]) := \left\{ v \in C^0((0,h]) : \lim_{t \to 0^+} t^r v(t) \text{ exists and is finite} \right\}.$$

Here $C^0((0,h])$ is the usual space of continuous functions on $(0,h]$. It turns out that the space $C_r^0([0,h])$ endowed with the norm

$$\|v\|_r := \max_{0 \le t \le h} t^r |v(t)|$$

is a Banach space.

The well-posedness has been discussed by Delbosco and Rodino in [3] and for a weighted fractional differential problem with a nonlinearity involving a nonlocal term of the form

$$f(t,u) + \int_0^t g(t,s,u(s)) ds \tag{7.2}$$

in [6]. But, it seems that the appropriate space to work on (introduced in [8]) is

$$C_{1-\alpha}^\alpha([0,h]) := \big\{ v \in C_{1-\alpha}^0([0,h]) : \text{ there exist } c \in \mathbb{R} \\ \text{and } v^* \in C_{1-\alpha}^0([0,h]) \text{ such that } v(t) = ct^{\alpha-1} + I^\alpha v^*(t) \big\}.$$

Sufficient conditions guaranteeing the existence of a fractional derivative $D^\alpha f$ and the representability of a function by a fractional integral of order α can be found in [24]. In particular, when

$$\int_0^t (t-s)^{-\alpha} f(s) ds \in AC([0,h])$$

(the space of absolutely continuous functions), then $D^\alpha f$ exists almost everywhere. Moreover, if $f(t) \in L^1(0,h)$ and $f_{1-\alpha} := I^{1-\alpha} f \in AC([0,h])$, then

$$f(t) = \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)} t^{\alpha-1} + I^\alpha D^\alpha f(t).$$

See [24, Theorem 2.4, p. 44].

Proposition 7.1. *If $\alpha > 1/2$, then the space $C_{1-\alpha}^\alpha([0,h])$ endowed with the norm*

$$\|v\|_{1-\alpha,\alpha} := \|v\|_{1-\alpha} + \|D^\alpha v\|_{1-\alpha}$$

is a Banach space.

In the space $C_{1-\alpha}^\alpha([0,h])$, it can be proved (see [8]) that the problem (7.1) is equivalent to the integral equation

$$u(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s)) ds. \tag{7.3}$$

Having this equation (7.3) we can use the argument in Section 2 to investigate the asymptotic behavior of solutions of (7.1). Some power type results have been established in [8]. In particular, we state

Theorem 7.2. *Suppose that $f(t,u)$ satisfies $(\mathbf{F})$, $\mu-(m-1)(1-\alpha)>0$ and $\alpha>1/2$. If $\lambda>0$ then $|u(t)|\leq Ct^{\alpha-1}$, $C>0$ on $[0,T]$ where T is fixed such that, for some (fixed and determined) constants K_i, $i=1,2,3$*

(1) *$\int_0^T \varphi^q(s)\exp(\varepsilon qs)ds\leq K_1$ for some $\varepsilon>0$, or*

(2) *(a) $T\leq 1$ and $\int_0^T\varphi^q(s)ds\leq K_2$, or*
(b) $T>1$ and

$$\int_0^T s^\gamma \exp\left(m\int_0^s b(\tau)d\tau\right)ds\leq K_3,$$

with

$$\gamma:=q\left[\frac{1}{p}+\mu-m(1-\alpha)\right]\quad \text{and}\quad b(t):=\frac{1}{m}\left(\frac{\left|\varphi'(t)\right|}{\varphi(t)}+\frac{1}{p}+\mu-(1-\alpha)m\right).$$

In this last case we assume that $\varphi(t)\geq d>0$ for all $t>0$.

The constant C is estimated by $2^{1+\frac{1}{q}\left(\frac{2-m}{m-1}\right)}|b|$ in (1) and (2) (a) and by $2d^{-1/m}|b|\varphi^{1/m}(0)\times \exp\left(\int_0^T b(\tau)d\tau\right)$ in the case (2) (b).

Corollary 7.3. *If instead of the assumption $(\mathbf{F})$ we have:*

$(\mathbf{F})'$ *$f(t,u)$ is continuous on $\mathbb{R}^+\times\mathbb{R}$ and is such that*

$$|f(t,u)|\leq t^\mu e^{-\sigma t}\varphi(t)\,|u|^m,\qquad \mu\geq 0,\ \sigma>0,\ m>1$$

then the solution of problem (7.1) exists globally and decays as a power function of non integer order on $\mathbb{R}^+$ provided that $\varphi\in L^q(\mathbb{R}^+)$ and $\|\varphi\|_q<\tilde{K}_1$.

For the same problem with the nonlinearity of the form (7.2), some other results have been proved in a recently submitted paper [7].

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