journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 2, Article 55, 24 pp.

GENERALIZED DUNKL-SOBOLEV SPACES OF EXPONENTIAL TYPE AND APPLICATIONS

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Received 22 March, 2008; accepted 23 May, 2009
Communicated by S.S. Dragomir

Dedicated to Khalifa Trimèche.

ABSTRACT. We study the Sobolev spaces of exponential type associated with the Dunkl-Bessel Laplace operator. Some properties including completeness and the imbedding theorem are proved. We next introduce a class of symbols of exponential type and the associated pseudo-differential-difference operators, which naturally act on the generalized Dunkl-Sobolev spaces of exponential type. Finally, using the theory of reproducing kernels, some applications are given for these spaces.

Key words and phrases: Dunkl operators, Dunkl-Bessel-Laplace operator, Generalized Dunkl-Sobolev spaces of exponential type, Pseudo differential-difference operators, Reproducing kernels.

2000 *Mathematics Subject Classification.* Primary 46F15. Secondary 46F12.

1. INTRODUCTION

The Sobolev space $W^{s,p}(\mathbb{R}^d)$ serves as a very useful tool in the theory of partial differential equations, which is defined as follows

$$W^{s,p}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d),\ (1 + ||\xi||^2)^{\frac{s}{p}} \mathcal{F}(u) \in L^p(\mathbb{R}^d) \right\}.$$

In this paper we consider the Dunkl-Bessel Laplace operator $\triangle_{k,\beta}$ defined by

$$\forall\, x = (x', x_{d+1}) \in \mathbb{R}^d \times]0, +\infty[, \qquad \triangle_{k,\beta} = \triangle_{k,x'} + L_{\beta, x_{d+1}}, \quad \beta \geq -\frac{1}{2},$$

where $\triangle_k$ is the Dunkl Laplacian on $\mathbb{R}^d$, and L_β is the Bessel operator on $]0, +\infty[$. We introduce the generalized Dunkl-Sobolev space of exponential type $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ by replacing

Thanks to the referee for his suggestions and comments.
090-08

$(1+||\xi||^2)^{\frac{s}{p}}$ by an exponential weight function defined as follows

$$W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+) = \Big\{ u \in \mathcal{G}'_*, e^{s||\xi||}\mathcal{F}_{D,B}(u) \in L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \Big\},$$

where $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ it is the Lebesgue space associated with the Dunkl-Bessel transform and $\mathcal{G}'_*$ is the topological dual of the Silva space. We investigate their properties such as the imbedding theorems and the structure theorems. In fact, the imbedding theorems mean that for $s > 0$, $u \in W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ can be analytically continued to the set $\{z \in \mathbb{C}^{d+1} / \ |\operatorname{Im} z| < s\}$. For the structure theorems we prove that for $s > 0$, $u \in W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ can be represented as an infinite sum of fractional Dunkl-Bessel Laplace operators of square integrable functions g, in other words,

$$u = \sum_{m\in\mathbb{N}} \frac{s^m}{m!}(-\triangle_{k,\beta})^{\frac{m}{2}} g.$$

We prove also that the generalized Dunkl-Sobolev spaces are stable by multiplication of the functions of the Silva spaces. As applications on these spaces, we study the action for the class of pseudo differential-difference operators and we apply the theory of reproducing kernels on these spaces. We note that special cases include: the classical Sobolev spaces of exponential type, the Sobolev spaces of exponential types associated with the Weinstein operator and the Sobolev spaces of exponential type associated with the Dunkl operators.

We conclude this introduction with a summary of the contents of this paper. In Section 2 we recall the harmonic analysis associated with the Dunkl-Bessel Laplace operator which we need in the sequel. In Section 3 we consider the Silva space $\mathcal{G}_*$ and its dual $\mathcal{G}'_*$. We study the action of the Dunkl-Bessel transform on these spaces. Next we prove two structure theorems for the space $\mathcal{G}'_*$. We define in Section 4 the generalized Dunkl-Sobolev spaces of exponential type $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and we give their properties. In Section 5 we give two applications on these spaces. More precisely, in the first application we introduce certain classes of symbols of exponential type and the associated pseudo-differential-difference operators of exponential type. We show that these pseudo-differential-difference operators naturally act on the generalized Sobolev spaces of exponential type. In the second, using the theory of reproducing kernels, some applications are given for these spaces.

2. PRELIMINARIES

In order to establish some basic and standard notations we briefly overview the theory of Dunkl operators and its relation to harmonic analysis. Main references are [3, 4, 5, 8, 16, 17, 19, 20, 21].

2.1. The Dunkl Operators.

Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle\cdot,\cdot\rangle$ and let $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$

orthogonal to α, i.e. for $x \in \mathbb{R}^d$,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R}\ \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R, reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. We fix a $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in R} H_\alpha$ and define a positive root system $R_+ = \{\alpha \in R \mid \langle \alpha, \beta\rangle > 0\}$. We normalize each $\alpha \in R_+$ as $\langle \alpha, \alpha\rangle = 2$. A function $k : R \longrightarrow \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. We introduce the index γ as

$$\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha).$$

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in R$. We denote by ω_k the weight function on $\mathbb{R}^d$ given by

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree 2γ, and by c_k the Mehta-type constant defined by

$$c_k = \left(\int_{\mathbb{R}^d} \exp(-||x||^2)\omega_k(x)\ dx\right)^{-1}.$$

We note that Etingof (cf. [6]) has given a derivation of the Mehta-type constant valid for all finite reflection group.

The Dunkl operators T_j, $j = 1, 2, \ldots, d$, on $\mathbb{R}^d$ associated with the positive root system R_+ and the multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d).$$

We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ for $f \in C^2(\mathbb{R}^d)$ by

$$\begin{aligned}\triangle_k f(x) &= \sum_{j=1}^d T_j^2 f(x)\\ &= \triangle f(x) + 2\sum_{\alpha \in R_+} k(\alpha)\left(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\right),\end{aligned}$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$ respectively. Then for each $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & j = 1, \ldots, d,\\ u(0, y) = 1 \end{cases}$$

admits a unique analytic solution $K(x, y)$, $x \in \mathbb{R}^d$, called the Dunkl kernel. This kernel has a holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$, (cf. [17] for the basic properties of K).

2.2. **Harmonic Analysis Associated with the Dunkl-Bessel Laplace Operator.** In this subsection we collect some notations and results on the Dunkl-Bessel kernel, the Dunkl-Bessel intertwining operator and its dual, the Dunkl-Bessel transform, and the Dunkl-Bessel convolution (cf. [12]).

In the following we denote by

- $\mathbb{R}^{d+1}_{+} = \mathbb{R}^d \times [0, +\infty[$.
- $x = (x_1, \dots, x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^{d+1}_{+}$.
- $C_*(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $C^p_*(\mathbb{R}^{d+1})$ the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$ the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{S}_*(\mathbb{R}^{d+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $D_*(\mathbb{R}^{d+1})$ the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathcal{S}'_*(\mathbb{R}^{d+1})$ the space of temperate distributions on $\mathbb{R}^{d+1}$, even with respect to the last variable. It is the topological dual of $\mathcal{S}_*(\mathbb{R}^{d+1})$.

We consider the Dunkl-Bessel Laplace operator $\triangle_{k,\beta}$ defined by

$$\forall\, x = (x', x_{d+1}) \,\in\, \mathbb{R}^d \times]0, +\infty[, \tag{2.1}$$
$$\triangle_{k,\beta} f(x) = \triangle_{k,x'} f(x', x_{d+1}) + \mathcal{L}_{\beta,x_{d+1}} f(x', x_{d+1}),\ f \in C^2_*(\mathbb{R}^{d+1}),$$

where $\triangle_k$ is the Dunkl-Laplace operator on $\mathbb{R}^d$, and $\mathcal{L}_\beta$ the Bessel operator on $]0, +\infty[$ given by

$$\mathcal{L}_\beta = \frac{d^2}{dx^2_{d+1}} + \frac{2\beta+1}{x_{d+1}} \frac{d}{dx_{d+1}}, \quad \beta > -\frac{1}{2}.$$

The Dunkl-Bessel kernel Λ is given by

$$\Lambda(x, z) = K(ix', z') j_\beta(x_{d+1} z_{d+1}), \quad (x, z) \in \mathbb{R}^{d+1} \times \mathbb{C}^{d+1}, \tag{2.2}$$

where $K(ix', z')$ is the Dunkl kernel and $j_\beta(x_{d+1} z_{d+1})$ is the normalized Bessel function. The Dunkl-Bessel kernel satisfies the following properties:

i) For all $z, t \in \mathbb{C}^{d+1}$, we have

$$\Lambda(z, t) = \Lambda(t, z); \quad \Lambda(z, 0) = 1 \quad \text{and} \quad \Lambda(\lambda z, t) = \Lambda(z, \lambda t), \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{2.3}$$

ii) For all $\nu \in \mathbb{N}^{d+1}, x \in \mathbb{R}^{d+1}$ and $z \in \mathbb{C}^{d+1}$, we have

$$|D^\nu_z \Lambda(x, z)| \leq ||x||^{|\nu|} \exp(||x||\, ||\operatorname{Im} z||), \tag{2.4}$$

where $D^\nu_z = \frac{\partial^\nu}{\partial z_1^{\nu_1} \cdots \partial z_{d+1}^{\nu_{d+1}}}$ and $|\nu| = \nu_1 + \cdots + \nu_{d+1}$. In particular

$$|\Lambda(x, y)| \leq 1, \quad \text{for all} \quad x, y \in \mathbb{R}^{d+1}. \tag{2.5}$$

The Dunkl-Bessel intertwining operator is the operator $\mathcal{R}_{k,\beta}$ defined on $C_*(\mathbb{R}^{d+1})$ by

(2.6) $$\mathcal{R}_{k,\beta}f(x', x_{d+1}) = \begin{cases} \dfrac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})} x_{d+1}^{-2\beta} \displaystyle\int_0^{x_{d+1}} (x_{d+1}^2 - t^2)^{\beta-\frac{1}{2}} V_k f(x', t)dt, & x_{d+1} > 0, \\ f(x', 0), & x_{d+1} = 0, \end{cases}$$

where V_k is the Dunkl intertwining operator (cf. [16]).

$\mathcal{R}_{k,\beta}$ is a topological isomorphism from $\mathcal{E}_*(\mathbb{R}^{d+1})$ onto itself satisfying the following transmutation relation

(2.7) $$\triangle_{k,\beta}(\mathcal{R}_{k,\beta}f) = \mathcal{R}_{k,\beta}(\triangle_{d+1}f), \quad \text{for all} \quad f \in \mathcal{E}_*(\mathbb{R}^{d+1}),$$

where $\triangle_{d+1} = \sum_{j=1}^{d+1} \partial_j^2$ is the Laplacian on $\mathbb{R}^{d+1}$.

The dual of the Dunkl-Bessel intertwining operator $\mathcal{R}_{k,\beta}$ is the operator ${}^t\mathcal{R}_{k,\beta}$ defined on $D_*(\mathbb{R}^{d+1})$ by: $\forall y = (y', y_{d+1}) \in \mathbb{R}^d \times [0, \infty[$,

(2.8) $${}^t\mathcal{R}_{k,\beta}(f)(y', y_{d+1}) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma\left(\beta+\frac{1}{2}\right)} \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta-\frac{1}{2}}\, {}^tV_k f(y', s)sds,$$

where ${}^tV_k$ is the dual Dunkl intertwining operator (cf. [20]).

${}^t\mathcal{R}_{k,\beta}$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself satisfying the following transmutation relation

(2.9) $${}^t\mathcal{R}_{k,\beta}(\triangle_{k,\beta}f) = \triangle_{d+1}({}^t\mathcal{R}_{k,\beta}f), \quad \text{for all} \quad f \in \mathcal{S}_*(\mathbb{R}^{d+1}).$$

We denote by $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ the space of measurable functions on $\mathbb{R}^{d+1}_+$ such that

$$||f||_{L^p_{k,\beta}(\mathbb{R}^{d+1}_+)} = \left(\int_{\mathbb{R}^{d+1}_+} |f(x)|^p d\mu_{k,\beta}(x)\, dx \right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad 1 \leq p < +\infty,$$
$$||f||_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} = \operatorname{ess} \sup_{x \in \mathbb{R}^{d+1}_+} |f(x)| < +\infty,$$

where $d\mu_{k,\beta}$ is the measure on $\mathbb{R}^{d+1}_+$ given by

$$d\mu_{k,\beta}(x', x_{d+1}) = \omega_k(x') x_{d+1}^{2\beta+1} dx' dx_{d+1}.$$

The Dunkl-Bessel transform is given for f in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ by

(2.10) $$\mathcal{F}_{D,B}(f)(y', y_{d+1}) = \int_{\mathbb{R}^{d+1}_+} f(x', x_{d+1}) \Lambda(-x, y) d\mu_{k,\beta}(x),$$
$$\text{for all} \quad y = (y', y_{d+1}) \in \mathbb{R}^{d+1}_+.$$

Some basic properties of this transform are the following:

i) For f in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$,

(2.11) $$||\mathcal{F}_{D,B}(f)||_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq ||f||_{L^1_{k,\beta}(\mathbb{R}^{d+1}_+)}.$$

ii) For f in $\mathcal{S}_*(\mathbb{R}^{d+1})$ we have

$$\mathcal{F}_{D,B}(\triangle_{k,\beta}f)(y) = -||y||^2\mathcal{F}_{D,B}(f)(y), \quad \text{for all} \quad y \in \mathbb{R}^{d+1}_+. \tag{2.12}$$

iii) For all $f \in \mathcal{S}(\mathbb{R}^{d+1}_*)$, we have

$$\mathcal{F}_{D,B}(f)(y) = \mathcal{F}_o \circ {}^t\mathcal{R}_{k,\beta}(f)(y), \quad \text{for all} \quad y \in \mathbb{R}^{d+1}_+, \tag{2.13}$$

where $\mathcal{F}_o$ is the transform defined by: $\forall\, y \in \mathbb{R}^{d+1}_+$,

$$\mathcal{F}_o(f)(y) = \int_{\mathbb{R}^{d+1}_+} f(x)e^{-i\langle y',x'\rangle}\cos(x_{d+1}y_{d+1})dx, \quad f \in D_*(\mathbb{R}^{d+1}). \tag{2.14}$$

iv) For all f in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, if $\mathcal{F}_{D,B}(f)$ belongs to $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, then

$$f(y) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\mathcal{F}_{D,B}(f)(x)\Lambda(x,y)d\mu_{k,\beta}(x), \quad a.e. \tag{2.15}$$

where

$$m_{k,\beta} = \frac{c_k^2}{4^{\gamma+\beta+\frac{d}{2}}(\Gamma(\beta+1))^2}. \tag{2.16}$$

v) For $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, if we define

$$\overline{\mathcal{F}_{D,B}}(f)(y) = \mathcal{F}_{D,B}(f)(-y),$$

then

$$\mathcal{F}_{D,B}\overline{\mathcal{F}_{D,B}} = \overline{\mathcal{F}_{D,B}}\mathcal{F}_D = m_{k,\beta}Id. \tag{2.17}$$

Proposition 2.1.

i) *The Dunkl-Bessel transform $\mathcal{F}_{D,B}$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself and for all f in $\mathcal{S}_*(\mathbb{R}^{d+1})$,*

$$\int_{\mathbb{R}^{d+1}_+}|f(x)|^2d\mu_{k,\beta}(x) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}|\mathcal{F}_{D,B}(f)(\xi)|^2d\mu_{k,\beta}(\xi). \tag{2.18}$$

ii) *In particular, the renormalized Dunkl-Bessel transform $f \to m^{\frac{1}{2}}_{k,\beta}\mathcal{F}_{D,B}(f)$ can be uniquely extended to an isometric isomorphism on $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$.*

By using the Dunkl-Bessel kernel, we introduce a generalized translation and a convolution structure. For a function $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$ the Dunkl-Bessel translation $\tau_y f$ is defined by the following relation:

$$\mathcal{F}_{D,B}(\tau_y f)(x) = \Lambda(x,y)\mathcal{F}_{D,B}(f)(x).$$

If $f \in \mathcal{E}_*(\mathbb{R}^{d+1})$ is radial with respect to the d first variables, i.e. $f(x) = F(||x'||, x_{d+1})$, then it follows that

$$\tau_y f(x) = \mathcal{R}_{k,\beta}\left[F\left(\sqrt{||x'||^2+||y'||^2+2\langle y',\cdot\rangle}, \sqrt{x^2_{d+1}+y^2_{d+1}+2y_{d+1}}\right)\right](x', x_{d+1}). \tag{2.19}$$

By using the Dunkl-Bessel translation, we define the Dunkl-Bessel convolution product $f *_{D,B} g$ of functions $f, g \in \mathcal{S}_*(\mathbb{R}^{d+1})$ as follows:

$$f *_{D,B} g(x) = \int_{\mathbb{R}_+^{d+1}} \tau_x f(-y) g(y) d\mu_{k,\beta}(y). \tag{2.20}$$

This convolution is commutative and associative and satisfies the following:

i) For all $f, g \in \mathcal{S}_*(\mathbb{R}_+^{d+1})$, $f *_{D,B} g$ belongs to $\mathcal{S}_*(\mathbb{R}_+^{d+1})$ and

$$\mathcal{F}_{D,B}(f *_{D,B} g)(y) = \mathcal{F}_{D,B}(f)(y)\mathcal{F}_{D,B}(g)(y). \tag{2.21}$$

ii) Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L^p_{k,\beta}(\mathbb{R}_+^{d+1})$ and $g \in L^q_{k,\beta}(\mathbb{R}_+^{d+1})$ is radial, then $f *_{D,B} g \in L^r_{k,\beta}(\mathbb{R}_+^{d+1})$ and

$$\|f *_{D,B} g\|_{L^r_{k,\beta}(\mathbb{R}_+^{d+1})} \leq \|f\|_{L^p_{k,\beta}(\mathbb{R}_+^{d+1})} \|g\|_{L^q_{k,\beta}(\mathbb{R}_+^{d+1})}. \tag{2.22}$$

3. STRUCTURE THEOREMS ON THE SILVA SPACE AND ITS DUAL

Definition 3.1. We denote by $\mathcal{G}_*$ or $\mathcal{G}_*(\mathbb{R}^{d+1})$ the set of all functions φ in $\mathcal{E}_*(\mathbb{R}^{d+1})$ such that for any $h, p > 0$

$$N_{p,h}(\varphi) = \sup_{\substack{x \in \mathbb{R}^{d+1} \\ \mu \in \mathbb{N}^{d+1}}} \left(\frac{e^{p||x||} |\partial^\mu \varphi(x)|}{h^{|\mu|} \mu!} \right)$$

is finite. The topology in $\mathcal{G}_*$ is defined by the above seminorms.

Lemma 3.1. *Let ϕ be in $\mathcal{G}_*$. Then for every $h, p > 0$*

$$N_{p,h}(\varphi) = \sup_{\substack{x \in \mathbb{R}^{d+1} \\ m \in \mathbb{N}}} \left(\frac{e^{p||x||} |\triangle^m_{k,\beta} \varphi(x)|}{h^m m!} \right).$$

Proof. We proceed as in Proposition 5.1 of [13], and by a simple calculation we obtain the result. □

Theorem 3.2. *The transform $\mathcal{F}_{D,B}$ is a topological isomorphism from $\mathcal{G}_*$ onto itself.*

Proof. From the relations (2.8), (2.9) and Lemma 3.1 we see that ${}^t\mathcal{R}_{k,\beta}$ is continuous from $\mathcal{G}_*$ onto itself. On the other hand, J. Chung et al. [1] have proved that the classical Fourier transform is an isomorphism from $\mathcal{G}_*$ onto itself. Thus from the relation (2.13) we deduce that $\mathcal{F}_{D,B}$ is continuous from $\mathcal{G}_*$ onto itself. Finally since $\mathcal{G}_*$ is included in $\mathcal{S}_*(\mathbb{R}^{d+1})$, and $\mathcal{F}_{D,B}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself, by (2.17) we obtain the result. □

We denote by $\mathcal{G}'_*$ or $\mathcal{G}'_*(\mathbb{R}^{d+1})$ the strong dual of the space $\mathcal{G}_*$.

Definition 3.2. The Dunkl-Bessel transform of a distribution S in $\mathcal{G}'_*$ is defined by

$$\langle \mathcal{F}_{D,B}(S), \psi \rangle = \langle S, \mathcal{F}_{D,B}(\psi) \rangle, \quad \psi \in \mathcal{G}_*.$$

The result below follows immediately from Theorem 3.2.

Corollary 3.3. *The transform $\mathcal{F}_{D,B}$ is a topological isomorphism from $\mathcal{G}'_*$ onto itself.*

Let τ be in $\mathcal{G}'_*$. We define $\triangle_{k,\beta}\tau$, by

$$\langle \triangle_{k,\beta}\tau, \psi\rangle = \langle \tau, \triangle_{k,\beta}\psi\rangle, \quad \text{for all} \quad \psi \in \mathcal{G}_*.$$

This functional satisfies the following property

$$\mathcal{F}_{D,B}(\triangle_{k,\beta}\tau) = -||y||^2\mathcal{F}_{D,B}(\tau). \tag{3.1}$$

Definition 3.3. The generalized heat kernel $\Gamma_{k,\beta}$ is given by

$$\Gamma_{k,\beta}(t,x,y) := \frac{2c_k}{\Gamma(\beta+1)(4t)^{\gamma+\beta+\frac{d}{2}+1}} e^{-\frac{||x||^2+||y||^2}{4t}} \Lambda\left(-i\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right); \quad x, y \in \mathbb{R}^{d+1}_+,\ t > 0.$$

The generalized heat kernel $\Gamma_{k,\beta}$ has the following properties:

Proposition 3.4. *Let x, y in $\mathbb{R}^{d+1}_+$ and $t > 0$. Then we have:*

i) $\Gamma_{k,\beta}(t,x,y) = \int_{\mathbb{R}^{d+1}_+} \exp(-t||\xi||^2)\Lambda(x,\xi)\Lambda(-y,\xi)d\mu_{k,\beta}(\xi).$

ii) $\int_{\mathbb{R}^{d+1}_+} \Gamma_{k,\beta}(t,x,y)d\mu_{k,\beta}(x) = 1.$

iii) *For fixed y in $\mathbb{R}^{d+1}_+$, the function $u(x,t) := \Gamma_{k,\beta}(t,x,y)$ solves the generalized heat equation:*

$$\triangle_{k,\beta}u(x,t) = \frac{\partial}{\partial t}u(x,t) \quad \text{on} \quad \mathbb{R}^{d+1}_+\times]0,+\infty[.$$

Definition 3.4. The generalized heat semigroup $(\mathcal{H}_{k,\beta}(t))_{t\geq 0}$ is the integral operator given for f in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ by

$$\mathcal{H}_{k,\beta}(t)f(x) := \begin{cases} \int_{\mathbb{R}^{d+1}_+} \Gamma_{k,\beta}(t,x,y)f(y)d\mu_{k,\beta}(y) & \text{if} \quad t > 0, \\ f(x) & \text{if} \quad t = 0. \end{cases}$$

From the properties of the generalized heat kernel we have

$$\mathcal{H}_{k,\beta}(t)f(x) := \begin{cases} f *_{D,B} p_t(x) & \text{if} \quad t > 0, \\ f(x) & \text{if} \quad t = 0, \end{cases} \tag{3.2}$$

where

$$p_t(y) = \frac{2c_k}{\Gamma(\beta+1)(4t)^{\gamma+\beta+\frac{d}{2}+1}} e^{-\frac{||y||^2}{4t}}.$$

Definition 3.5. A function f defined on $\mathbb{R}^{d+1}_+$ is said to be of exponential type if there are constants $k, C > 0$ such that for every $x \in \mathbb{R}^{d+1}_+$

$$|f(x)| \leq \exp k||x||.$$

The following lemma will be useful later. For the details of the proof we refer to Komatsu [9]:

Lemma 3.5. *For any $L > 0$ and $\varepsilon > 0$ there exist a function $v \in D(\mathbb{R})$ and a differential operator $P(\frac{d}{dt})$ of infinite order such that*

$$\operatorname{supp} v \subset [0,\varepsilon], \quad |v(t)| \le C \exp\left(-\frac{L}{t}\right), \quad 0 < t < \infty; \tag{3.3}$$

$$P\left(\frac{d}{dt}\right) = \sum_{k=0}^{\infty} a_k \left(\frac{d}{dt}\right)^k, \quad |a_k| \le C_1 \frac{L_1^k}{k!^2}, \quad 0 < L_1 < L; \tag{3.4}$$

$$P\left(\frac{d}{dt}\right) v(t) = \delta + \omega(t), \tag{3.5}$$

where $\omega \in D(\mathbb{R}), \operatorname{supp} \omega \subset [\frac{\varepsilon}{2}, \varepsilon]$.

Here we note that $P(\triangle_{k,\beta})$ is a local operator where $\triangle_{k,\beta}$ is a Dunkl-Bessel Laplace operator.

Now we are in a position to state and prove one of the main theorems in this section.

Theorem 3.6. *If $u \in \mathcal{G}'_*$ then there exists a differential operator $P(\frac{d}{dt})$ such that for some $C > 0$ and $L > 0$,*

$$P\left(\frac{d}{dt}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{d}{dt}\right)^n, \quad |a_n| \le C \frac{L^n}{n!^2},$$

and there are a continuous function g of exponential type and an entire function h of exponential type in $\mathbb{R}_+^{d+1}$ such that

$$u = P(\triangle_{k,\beta}) g(x) + h(x). \tag{3.6}$$

Proof. Let $U(x,t) = \langle u_y, \Gamma_{k,\beta}(t,x,y)\rangle$. Since p_t belongs to $\mathcal{G}_*$ for each $t > 0$, $U(x,t)$ is well defined and real analytic in $(\mathbb{R}_+^{d+1})_x$ for each $t > 0$.

Furthermore, $U(x,t)$ satisfies

$$(\partial_t - \triangle_{k,\beta}) U(x,t) = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}_+^{d+1} \times]0,\infty[. \tag{3.7}$$

Here $u \in \mathcal{G}'_*$ means that for some $k > 0$ and $h > 0$

$$|\langle u, \phi\rangle| \le C \sup_{x,\alpha} \frac{|\partial^\alpha \phi(x)| \exp k||x||}{h^{|\alpha|}\alpha!}, \quad \phi \in \mathcal{G}_*. \tag{3.8}$$

By Cauchy's inequality and relations (2.4) and (3.8) we obtain, for $t > 0$

$$|U(x,t)| \le C' \exp k' \left[||x|| + t + \frac{1}{t} \right]$$

for some $C' > 0$ and $k' > 0$. If we restrict this inequality on the strip $0 < t < \varepsilon$ then it follows that

$$|U(x,t)| \le C \exp k \left[||x|| + \frac{1}{t} \right], \quad 0 < t < \varepsilon$$

for some constants $C > 0$ and $k > 0$. Now let

$$G(y,t) = \int_{\mathbb{R}_+^{d+1}} \Gamma_{k,\beta}(t,x,y) \phi(x) d\mu_{k,\beta}(x), \quad \phi \in \mathcal{G}_*.$$

Moreover, we can easily see that

$$G(\cdot,t) \to \phi \quad \text{in} \quad \mathcal{G}_* \text{ as } t \to 0^+ \tag{3.9}$$

and

$$\int_{\mathbb{R}^{d+1}_+} U(x,t)\phi(x)d\mu_{k,\beta}(x) = \langle u_y, G(y,t)\rangle. \tag{3.10}$$

Then it follows from (3.9) and (3.10) that

$$\lim_{t\to 0^+} U(x,t) = u \quad \text{in } \mathcal{G}'_*. \tag{3.11}$$

Now choose functions u, w and a differential operator of infinite order as in Lemma 3.5. Let

$$\tilde{U}(x,t) = \int_0^\infty U(x,t+s)v(s)ds. \tag{3.12}$$

Then by taking $\varepsilon = 2$ and $L > k$ in Lemma 3.5 we have

$$|\tilde{U}(x,t)| \le C' \exp k\big(||x|| + t\big), \quad t \ge 0. \tag{3.13}$$

Therefore, $\tilde{U}(x,t)$ is a continuous function of exponential type in

$$\mathbb{R}^{d+1}_+ \times [0,\infty[= \Big\{(x,t) : x \in \mathbb{R}^{d+1}_+, t \ge 0\Big\}.$$

Moreover, $\tilde{U}$ satisfies

$$(\partial_t - \triangle_{k,\beta})\tilde{U}(x,t) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ \times]0,\infty[. \tag{3.14}$$

Hence if we set $g(x) = \tilde{U}(x,0)$ then g is also a continuous function of exponential type, so that g belongs to $\mathcal{G}'_*$.

Using (3.5) in Lemma 3.5, we obtain for $t > 0$

$$\begin{aligned} P(-\triangle_{k,\beta})\tilde{U}(x,t) &= P\left(-\frac{d}{dt}\right)\tilde{U}(x,t) \\ &= U(x,t) + \int_0^\infty U(x,t+s)w(s)ds. \end{aligned} \tag{3.15}$$

If we set $h(x) = -\int_0^\infty U(x,s)w(s)ds$ then h is an entire function of exponential type. As $t \to 0^+$, (3.15) becomes

$$u = P(-\triangle_{k,\beta})g(x) + h(x)$$

which completes the proof by replacing the coefficients a_n of P by $(-1)^n a_n$. □

Theorem 3.7. *Let $U(x,t)$ be an infinitely differentiable function in $\mathbb{R}^{d+1}_+ \times]0,\infty[$ satisfying the conditions:*

i) $(\partial_t - \triangle_{k,\beta})U(x,t) = 0$ *in* $\mathbb{R}^{d+1}_+ \times]0,\infty[$.

ii) *There exist $k > 0$ and $C > 0$ such that*

$$|U(x,t)| \le C \exp k\left(||x|| + \frac{1}{t}\right), 0 < t < \varepsilon, \quad x \in \mathbb{R}^{d+1}_+ \tag{3.16}$$

for some $\varepsilon > 0$. Then there exists a unique element $u \in \mathcal{G}'_*$ such that

$$U(x,t) = \langle u_y, \Gamma_{k,\beta}(t,x,y)\rangle, \quad t > 0$$

and

$$\lim_{t\to 0^+} U(x,t) = u \quad \text{in} \quad \mathcal{G}'_*.$$

Proof. Consider a function, as in (3.12)

$$\tilde{U}(x,t) = \int_0^\infty U(x,t+s)v(s)ds.$$

Then it follows from (3.13), (3.14) and (3.15) that $\tilde{U}(x,t)$ and $H(x,t)$ are continuous on $\mathbb{R}^{d+1}_+ \times [0,\infty[$ and

$$U(x,t) = P(-\triangle_{k,\beta})\tilde{U}(x,t) + H(x,t), \tag{3.17}$$

where

$$H(x,t) = -\int_0^\infty U(x,t+s)w(s)ds.$$

Furthermore, $g(x) = \tilde{U}(x,0)$ and $h(x) = H(x,0)$ are continuous functions of exponential type on $\mathbb{R}^{d+1}_+$. Define u as

$$u = P(-\triangle_{k,\beta})g(x) + h(x).$$

Then since $P(-\triangle_{k,\beta})$ is a local operator, u belongs to $\mathcal{G}'_*$ and

$$\lim_{t\to 0^+} U(x,t) = u \quad \text{in} \quad \mathcal{G}'_*.$$

Now define the generalized heat kernels for $t > 0$ as

$$A(x,t) = (g \ *_{D,B} \ p_t)(x) = \int_{\mathbb{R}^{d+1}_+} g(y)\Gamma_{k,\beta}(t,x,y)d\mu_{k,\beta}(y)$$

and

$$B(x,t) = (h \ *_{D,B} \ p_t)(x) = \int_{\mathbb{R}^{d+1}_+} h(y)\Gamma_{k,\beta}(t,x,y)d\mu_{k,\beta}(y).$$

Then it is easy to show that $A(x,t)$ and $B(x,t)$ converge locally uniformly to $g(x)$ and $h(x)$ respectively so that they are continuous on $\mathbb{R}^{d+1}_+ \times [0,\infty[$, $A(x,0) = g(x)$, and $B(x,0) = h(x)$. Now let $V(x,t) = \tilde{U}(x,t) - A(x,t)$ and $W(x,t) = H(x,t) - B(x,t)$. Then, since g and h are of exponential type, $V(\cdot,t)$ and $W(\cdot,t)$ are continuous functions of exponential type and $V(x,0) = 0, W(x,0) = 0$. Then by the uniqueness theorem of the generalized heat equations we obtain that

$$\tilde{U}(x,t) = \big(g \ *_{D,B} \ p_t\big)(x)$$

and

$$H(x,t) = \big(h \ *_{D,B} \ p_t\big)(x).$$

It follows from these facts and (3.17) that

$$\begin{aligned} u \ *_{D,B} \ p_t &= \Big[P(-\triangle_{k,\beta})g + h\Big] *_{D,B} \ p_t \\ &= P(-\triangle_{k,\beta})\tilde{U}(\cdot,t) + H(\cdot,t) \\ &= U(\cdot,t). \end{aligned}$$

Now to prove the uniqueness of existence of such $u \in \mathcal{G}'_*$ we assume that there exist $u, v \in \mathcal{G}'_*$ such that

$$U(x,t) = \big(u \ *_{D,B} \ p_t\big)(x) = \big(v \ *_{D,B} \ p_t\big)(x).$$

Then

$$\mathcal{F}_{D,B}(u)\mathcal{F}_{D,B}(p_t) = \mathcal{F}_{D,B}(v)\mathcal{F}_{D,B}(p_t)$$

which implies that $\mathcal{F}_{D,B}(u) = \mathcal{F}_{D,B}(v)$, since $\mathcal{F}_{D,B}(p_t) \neq 0$. However, since the Dunkl-Bessel transformation is an isomorphism we have $u = v$, which completes the proof. □

4. The Generalized Dunkl-Sobolev Spaces of Exponential Type

Definition 4.1. Let s be in $\mathbb{R}$, $1 \leq p < \infty$. We define the space $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ by

$$\Big\{u \in \mathcal{G}'_* : e^{s||\xi||}\mathcal{F}_{D,B}(u) \in L^p_{k,\beta}(\mathbb{R}^{d+1}_+)\Big\}.$$

The norm on $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ is given by

$$||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} = \left(m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} e^{ps||\xi||}|\mathcal{F}_{D,B}(u)(\xi)|^p d\mu_{k,\beta}(\xi)\right)^{\frac{1}{p}}.$$

For $p = 2$ we provide this space with the scalar product

$$\langle u, v\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} e^{2s||\xi||}\mathcal{F}_{D,B}(u)(\xi)\overline{\mathcal{F}_{D,B}(v)(\xi)}d\mu_{k,\beta}(\xi), \tag{4.1}$$

and the norm

$$||u||^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} = \langle u, u\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}}. \tag{4.2}$$

Proposition 4.1.

i) *Let $1 \leq p < +\infty$. The space $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ provided with the norm $\|\cdot\|_{W^{s,p}_{\mathcal{G}_*,k,\beta}}$ is a Banach space.*

ii) *We have*

$$W^{0,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+) = L^2_{k,\beta}(\mathbb{R}^{d+1}_+).$$

iii) *Let $1 \leq p < +\infty$ and s_1, s_2 in $\mathbb{R}$ such that $s_1 \geq s_2$ then*

$$W^{s_1,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+) \hookrightarrow W^{s_2,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+).$$

Proof. i) It is clear that the space $L^p(\mathbb{R}^{d+1}_+, e^{ps||\xi||}d\mu_{k,\beta}(\xi))$ is complete and since $\mathcal{F}_{D,B}$ is an isomorphism from $\mathcal{G}'_*$ onto itself, $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ is then a Banach space.

The results ii) and iii) follow immediately from the definition of the generalized Dunkl-Sobolev space of exponential type. □

Proposition 4.2. *Let $1 \leq p < +\infty$, and s_1, s, s_2 be three real numbers satisfying $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a nonnegative constant C_ε such that for all u in $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$*

$$||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} \leq C_\varepsilon ||u||_{W^{s_1,p}_{\mathcal{G}_*,k,\beta}} + \varepsilon ||u||_{W^{s_2,p}_{\mathcal{G}_*,k,\beta}}. \tag{4.3}$$

Proof. We consider $s = (1-t)s_1 + ts_2$, (with $t \in]0,1[$). Moreover it is easy to see

$$||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} \leq ||u||^{1-t}_{W^{s_1,p}_{\mathcal{G}_*,k,\beta}} ||u||^{t}_{W^{s_2,p}_{\mathcal{G}_*,k,\beta}}.$$

Thus

$$\begin{aligned} ||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} &\leq \left(\varepsilon^{-\frac{t}{1-t}} ||u||_{W^{s_1,p}_{\mathcal{G}_*,k,\beta}}\right)^{1-t} \left(\varepsilon ||u||_{W^{s_2,p}_{\mathcal{G}_*,k,\beta}}\right)^{t} \\ &\leq \varepsilon^{-\frac{t}{1-t}} ||u||_{W^{s_1,p}_{\mathcal{G}_*,k,\beta}} + \varepsilon ||u||_{W^{s_2,p}_{\mathcal{G}_*,k,\beta}}. \end{aligned}$$

Hence the proof is completed for $C_\varepsilon = \varepsilon^{-\frac{t}{1-t}}$. □

Proposition 4.3. *For s in $\mathbb{R}$, $1 \leq p < \infty$ and m in $\mathbb{N}$, the operator $\triangle^m_{k,\beta}$ is continuous from $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ into $W^{s-\varepsilon,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ for any $\varepsilon > 0$.*

Proof. Let u be in $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, and m in $\mathbb{N}$. From (3.1) we have

$$\int_{\mathbb{R}^{d+1}_+} e^{p(s-\varepsilon)||\xi||} |\mathcal{F}_{D,B}(\triangle^m_{k,\beta} u)(\xi)|^p d\mu_{k,\beta}(\xi) = \int_{\mathbb{R}^{d+1}_+} e^{p(s-\varepsilon)||\xi||} ||\xi||^{2mp} |\mathcal{F}_{D,B}(u)(\xi)|^p d\mu_{k,\beta}(\xi).$$

As $\sup_{\xi \in \mathbb{R}^{d+1}_+} ||\xi||^{2mp} e^{-\varepsilon||\xi||} < \infty$, for every $m \in \mathbb{N}$ and $\varepsilon > 0$

$$\int_{\mathbb{R}^{d+1}_+} e^{p(s-\varepsilon)||\xi||} |\mathcal{F}_{D,B}(\triangle^m_{k,\beta} u)(\xi)|^p d\mu_{k,\beta}(\xi) \leq \int_{\mathbb{R}^{d+1}_+} e^{ps||\xi||} |\mathcal{F}_{D,B}(u)(\xi)|^p d\mu_{k,\beta}(\xi) < +\infty.$$

Then $\triangle^m_{k,\beta} u$ belongs to $W^{s-\varepsilon,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, and

$$||\triangle^m_{k,\beta} u||_{W^{s-\varepsilon,p}_{\mathcal{G}_*,k,\beta}} \leq ||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}}.$$

□

Definition 4.2. We define the operator $(-\triangle_{k,\beta})^{\frac{1}{2}}$ by $\forall x \in \mathbb{R}^{d+1}_+$,

$$(-\triangle_{k,\beta})^{\frac{1}{2}} u(x) = m_{k,\beta} \int_{\mathbb{R}^{d+1}_+} \Lambda(x,\xi) ||\xi|| \mathcal{F}_{D,B}(u)(\xi) d\mu_{k,\beta}(\xi), \quad u \in \mathcal{S}_*(\mathbb{R}^{d+1}).$$

Proposition 4.4. *Let $P((-\triangle_{k,\beta})^{\frac{1}{2}}) = \sum_{m\in\mathbb{N}} a_m (-\triangle_{k,\beta})^{\frac{m}{2}}$ be a fractional Dunkl-Bessel Laplace operators of infinite order such that there exist positive constants C and r such that*

$$|a_m| \leq C \frac{r^m}{m!}, \quad \text{for all} \quad m \in \mathbb{N}. \tag{4.4}$$

Let $1 \leq p < +\infty$ and s in $\mathbb{R}$. If an element u is in $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, then $P((-\triangle_{k,\beta})^{\frac{1}{2}})u$ belongs to $W^{s-r,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, and there exists a positive constant C such that

$$||P((-\triangle_{k,\beta})^{\frac{1}{2}})u||_{W^{s-r,p}_{\mathcal{G}_*,k,\beta}} \leq C ||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}}.$$

Proof. The condition (4.4) gives that

$$|P(||\xi||)| \leq C\, \exp(r\, ||\xi||).$$

Thus the result is immediate. □

Proposition 4.5. *Let* $1 \leq p < +\infty$*,* t, s *in* $\mathbb{R}$*. The operator* $\exp(t(-\triangle_{k,\beta})^{\frac{1}{2}})$ *defined by*

$$\forall x \in \mathbb{R}^{d+1}_+,\ \exp(t(-\triangle_{k,\beta})^{\frac{1}{2}})u(x) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\Lambda(x,\xi)e^{t||\xi||}\mathcal{F}_{D,B}(u)(\xi)d\mu_{k,\beta}(\xi),\ \ u \in \mathcal{G}_*.$$

is an isomorphism from $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ *onto* $W^{s-t,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$*.*

Proof. For u in $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ it is easy to see that

$$\|\exp(t(-\triangle_{k,\beta})^{\frac{1}{2}})u\|_{W^{s-t,p}_{\mathcal{G}_*,k,\beta}} = ||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}},$$

and thus we obtain the result. □

Proposition 4.6. *The dual of* $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ *can be identified as* $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$*. The relation of the identification is given by*

$$\langle u, v\rangle_{k,\beta} = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\mathcal{F}_{D,B}(u)(\xi)\overline{\mathcal{F}_{D,B}(v)(\xi)}d\mu_{k,\beta}(\xi), \tag{4.5}$$

with u *in* $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ *and* v *in* $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$*.*

Proof. Let u be in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and v in $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$. The Cauchy-Schwartz inequality gives

$$|\langle u, v\rangle_{k,\beta}| \leq ||u||_{W^{s,2}_{\mathcal{G}_*,k,\beta}}||v||_{W^{-s,2}_{\mathcal{G}_*,k,\beta}}.$$

Thus for v in $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ fixed we see that $u \longmapsto \langle u, v\rangle_{k,\beta}$ is a continuous linear form on $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ whose norm does not exceed $||v||_{W^{-s,2}_{\mathcal{G}_*,k,\beta}}$. Taking the $u_0 = \mathcal{F}^{-1}_{D,B}\left(e^{-2s||\xi||}\mathcal{F}_{D,B}(v)\right)$ element of $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, we obtain $\langle u_0, v\rangle_{k,\beta} = ||v||_{W^{-s,2}_{\mathcal{G}_*,k,\beta}}$.

Thus the norm of $u \longmapsto \langle u, v\rangle_{k,\beta}$ is equal to $||v||_{W^{-s,2}_{\mathcal{G}_*,k,\beta}}$, and we have then an isometry:

$$W^{-s,2}_{\mathcal{G}_*,k,\beta}\left(\mathbb{R}^{d+1}_+\right) \longrightarrow \left(W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)\right)'.$$

Conversely, let L be in $\left(W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)\right)'$. By the Riesz representation theorem and (4.1) there exists w in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ such that

$$\begin{aligned} L(u) &= \langle u, w\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \\ &= m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}e^{2s||\xi||}\mathcal{F}_{D,B}(u)(\xi)\overline{\mathcal{F}_{D,B}(w)(\xi)}d\mu_{k,\beta}(\xi), \quad \text{for all } u \in W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+). \end{aligned}$$

If we set $v = \mathcal{F}^{-1}_{D}\left(e^{2s||\xi||}\mathcal{F}_{D,B}(w)\right)$ then v belongs to $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and $L(u) = \langle u, v\rangle_{k,\beta}$ for all u in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, which completes the proof. □

Proposition 4.7. *Let $s > 0$. Then every u in $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ can be represented as an infinite sum of fractional Dunkl-Bessel Laplace operators of square integrable function g, in other words,*

$$u = \sum_{m\in\mathbb{N}} \frac{s^m}{m!}(-\triangle_{k,\beta})^{\frac{m}{2}} g.$$

Proof. If u in $W^{-s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ then by definition

$$e^{-s||\xi||}\mathcal{F}_{D,B}(u)(\xi) \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+),$$

which Proposition 2.1 ii) implies that

$$\mathcal{F}_{D,B}(g)(\xi) = \frac{\mathcal{F}_{D,B}(u)(\xi)}{\sum_{m\in\mathbb{N}} \frac{s^m}{m!}||\xi||^m} \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+).$$

Hence, we have

$$\begin{aligned}\mathcal{F}_{D,B}(u) &= \sum_{m\in\mathbb{N}} \frac{s^m}{m!}||\xi||^m \mathcal{F}_{D,B}(g), \quad \text{in} \quad \mathcal{G}'_* \\ &= \sum_{m\in\mathbb{N}} \frac{s^m}{m!}\mathcal{F}_{D,B}\left((-\triangle_{k,\beta})^{\frac{m}{2}} g\right), \quad \text{in} \quad \mathcal{G}'_*\end{aligned}$$

This completes the proof. □

Proposition 4.8. *Let $1 \le p < +\infty$. Every u in $W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ is a holomorphic function in the strip $\{z \in \mathbb{C}^{d+1},\ ||\operatorname{Im} z|| < s\}$ for $s > 0$.*

Proof. Let

$$u(z) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} \Lambda(z,\xi)\mathcal{F}_{D,B}(u)(\xi)d\mu_{k,\beta}(\xi), \quad z = x + iy.$$

From (2.4), for each μ in $\mathbb{N}^{d+1}$ we have

$$\left|D^\mu_z\Big(\Lambda(z,\xi)\mathcal{F}_{D,B}(u)(\xi)\Big)\right| \le ||\xi||^{|\mu|}e^{||y||\,||\xi||}|\mathcal{F}_{D,B}(u)(\xi)|.$$

On the other hand from the Cauchy-Schwartz inequality we have

$$\int_{\mathbb{R}^{d+1}_+} ||\xi^\mu||e^{||y||\,||\xi||}|\mathcal{F}_{D,B}(u)(\xi)|d\mu_{k,\beta}(\xi) \le \left(\int_{\mathbb{R}^{d+1}_+} ||\xi||^{q|\mu|}e^{q||\xi||(||y||-s)}d\mu_{k,\beta}(\xi)\right)^{\frac{1}{q}} ||u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}}.$$

Since the integral in the last part of the above inequality is integrable if $||y|| < s$, the result follows by the theorem of holomorphy under the integral sign. □

Notations. Let m be in $\mathbb{N}$. We denote by:

- $\mathcal{E}'_m(\mathbb{R}^{d+1}_+)$ the space of distributions on $\mathbb{R}^{d+1}_+$ with compact support and order less than or equal to m.
- $\mathcal{E}'_{\exp,m}(\mathbb{R}^{d+1}_+)$ the space of distributions u in $\mathcal{E}'_m(\mathbb{R}^{d+1}_+)$ such that there exists a positive constant C such that

$$|\mathcal{F}_{D,B}(u)(\xi)| \le Ce^{m||\xi||}.$$

Proposition 4.9.

i) *Let* $1 \leq p < +\infty$. *For* s *in* $\mathbb{R}$ *such that* $s < -m$, *we have*

$$\mathcal{E}'_{\exp,m}(\mathbb{R}^{d+1}_+) \subset W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+).$$

ii) *Let* $1 \leq p < +\infty$. *We have, for* $s < 0$ *and* m *an integer,*

$$\mathcal{E}'_m(\mathbb{R}^{d+1}_+) \subset W^{s,p}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+).$$

Proof. The proof uses the same idea as Theorem 3.12 of [13]. □

Theorem 4.10. *Let* ψ *be in* $\mathcal{G}_*$. *For all* s *in* $\mathbb{R}$, *the mapping* $u \mapsto \psi u$ *from* $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ *into itself is continuous.*

Proof. Firstly we assume that $s > 0$.

It is easy to see that

$$||v||^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \leq \sum_{j=0}^{\infty} \frac{(2s)^j}{j!} ||v||^2_{\dot{H}^{\frac{j}{2}}_{k,\beta}(\mathbb{R}^{d+1}_+)},$$

where $\|\cdot\|_{\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)}$ designates the norm associated to the homogeneous Dunkl-Bessel-Sobolev space defined by

$$||v||^2_{\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)} = \int_{\mathbb{R}^{d+1}_+} ||\xi||^{2s} |\mathcal{F}_{D,B}(v)(\xi)|^2 d\mu_{k,\beta}(\xi).$$

On the other hand, proceeding as in Proposition 4.1 of [14] we prove that if u, v belong to $\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+) \bigcap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$, $s > 0$ then $uv \in \dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)$ and

$$||uv||_{\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq C \left[||u||_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} ||v||_{\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)} + ||v||_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} ||u||_{\dot{H}^s_{k,\beta}(\mathbb{R}^{d+1}_+)} \right].$$

Thus from this we deduce that for $s > 0$

$$\begin{aligned} ||\varphi u||^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} &\leq \sum_{j=0}^{\infty} \frac{(2s)^j}{j!} ||\varphi u||^2_{\dot{H}^{\frac{j}{2}}_{k,\beta}(\mathbb{R}^{d+1}_+)} \\ &\leq C \left[||\varphi||^2_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} ||u||^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} + ||u||^2_{L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)} ||\varphi||^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \right] < +\infty. \end{aligned}$$

For $s = 0$ the result is immediate.

For $s < 0$ the result is obtained by duality. □

5. APPLICATIONS

5.1. **Pseudo-differential-difference operators of exponential type.** The pseudo-differential-difference operator $A(x, \triangle_{k,\beta})$ associated with the symbol $a(x,\xi) := A(x, -||\xi||^2)$ is defined by

$$\text{(5.1)} \qquad \big(A(x, \triangle_{k,\beta})u\big)(x) = m_{k,\beta} \int_{\mathbb{R}^{d+1}_+} \Lambda(x,\xi) a(x,\xi) \mathcal{F}_{D,B}(u)(\xi) d\mu_{k,\beta}(\xi), \ u \in \mathcal{G}_*,$$

where $a(x,\xi)$ belongs to the class $S^r_{\exp}$, $r \geq 0$, defined below:

Definition 5.1. The function $a(x,\xi)$ is said to be in $S^r_{\exp}$ if and only if $a(x,\xi)$ belongs to $C^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})$ and for each compact set $K\subset\mathbb{R}^{d+1}_+$ and each μ,ν in $\mathbb{N}^{d+1}$, there exists a constant $C_K=C_{\mu,\nu,K}$ such that the estimate

$$|D^\mu_\xi D^\nu_x a(x,\xi)|\le C_K\,\exp(r\,||\xi||),\quad \text{for all}\quad (x,\xi)\in K\times\mathbb{R}^{d+1}_+$$

hold true.

Proposition 5.1. *Let $A(x,\triangle_{k,\beta})$ be the pseudo-differential-difference operator associated with the symbol $a(x,\xi):=A(x,-||\xi||^2)$. If $a(x,\xi)\in S^r_{\exp}$ then $A(x,\triangle_{k,\beta})$ in (5.1) is a well-defined mapping of $\mathcal{G}_*$ into $\mathcal{E}_*(\mathbb{R}^{d+1})$.*

Proof. For any compact set $K\subset\mathbb{R}^{d+1}_+$, we have

$$|a(x,\xi)|\le C_K\,\exp(r\,||\xi||),\ \text{for all}\,(x,\xi)\in K\times\mathbb{R}^{d+1}_+.$$

On the other hand since u is in $\mathcal{G}_*$ the Cauchy-Schwartz inequality gives that

$$\int_{\mathbb{R}^{d+1}_+}|\Lambda(x,\xi)a(x,\xi)\mathcal{F}_{D,B}(u)(\xi)|d\mu_{k,\beta}(\xi)\le C_K||u||_{W^{s,2}_{\mathcal{G},k,\beta}}\left(\int_{\mathbb{R}^{d+1}_+}e^{-2(s-r)||\xi||}d\mu_{k,\beta}(\xi)\right)^{\frac12}$$

is integrable for $s>r$. This prove the existence and the continuity of $(A(x,\triangle_{k,\beta})u)(x)$ for all x in $\mathbb{R}^{d+1}_+$. Finally the result follows by using Leibniz formula. □

Now we consider the symbol which belongs to the class $S^{r,l}_{\exp,rad}$ defined below:

Definition 5.2. Let r,l in $\mathbb{R}$ be real numbers with $l>0$. The function $a(x,\xi)$ is said to be in $S^{r,l}_{\exp,rad}$ if and only if $a(x,\xi)$ is in $C^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})$, radial with respect to the first $d+1$ variables and for each $L>0$, and for each μ,ν in $\mathbb{N}^{d+1}$, there exists a constant $C=C_{r,\mu,\nu}$ such that the estimate

$$|D^\mu_\xi D^\nu_x a(x,\xi)|\le CL^{|\mu|}\mu!\exp(r\,||\xi||-l\,||x||)$$

hold true.

To obtain some deep and interesting results we need the following alternative form of $A(x,\triangle_{k,\beta})$.

Lemma 5.2. *Let $A(x,\triangle_{k,\beta})$ be the pseudo-differential-difference operator associated with the symbol $a(x,\xi):=A(x,-||\xi||^2)$. If $a(x,\xi)$ is in $S^{r,l}_{\exp,rad}$ then $A(x,\triangle_{k,\beta})$ in (5.1) is given by:*

$$\begin{aligned}&\big(A(x,\triangle_{k,\beta})u\big)(x)\\&\quad=m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\Lambda(x,\xi)\left[\int_{\mathbb{R}^{d+1}_+}\tau_{-\eta}\big(\mathcal{F}_{D,B}(a)(\cdot,\eta)\big)(\xi)\mathcal{F}_{D,B}(u)(\eta)d\mu_{k,\beta}(\eta)\right]d\mu_{k,\beta}(\xi)\end{aligned}$$

for all $u\in\mathcal{G}_*$ where all the involved integrals are absolutely convergent.

Proof. When we proceed as [10] we see that for all $L>0$ there exist $C>0$ and $0<\tau<(Ld)^{-1}$ such that

$$|\mathcal{F}_{D,B}(a)(\xi,\eta)|\le C\exp(r\,||\eta||-\tau\,||\xi||). \tag{5.2}$$

On the other hand since u in $\mathcal{G}_*$, we have

$$|\mathcal{F}_{D,B}(u)(\eta)| \leq C\exp(-t\,||\eta||),\ \forall\, t>0.$$

Thus from the positivity for the Dunkl-Bessel translation operator for the radial functions we obtain

$$\left|\tau_{-\eta}\big(\mathcal{F}_{D,B}(a)(\cdot,\eta)\big)(\xi)\mathcal{F}_{D,B}(u)(\eta)\right| \leq C\exp(-(t-r)\,||\eta||)\tau_{-\eta}(\exp(-\tau\ \|\cdot\|))(\xi).$$

Moreover it is clear that the function $\eta \mapsto \exp(-(t-r)\,||\eta||)\tau_{-\eta}(\exp(-\tau\ \|\cdot\|))(\xi)$ belongs to $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ for $t>r$, $\tau>0$. So that

$$\int_{\mathbb{R}^{d+1}_+}\left|\tau_{-\eta}\big(\mathcal{F}_{D,B}(a)(\cdot,\eta)\big)(\xi)\mathcal{F}_{D,B}(u)(\eta)\right| d\mu_{k,\beta}(\eta)$$
$$\leq C\int_{\mathbb{R}^{d+1}_+}\exp(-(t-r)\,||\eta||)\tau_{-\eta}(\exp(-\tau\,||\cdot||))(\xi)d\mu_{k,\beta}(\eta).$$

The right-hand side is a Dunkl-Bessel convolution product of two integrable functions and hence is an integrable function on $\mathbb{R}^{d+1}_+$. Therefore the function

$$\xi \mapsto \int_{\mathbb{R}^{d+1}_+}\tau_{-\eta}(\mathcal{F}_{D,B}(a)(\cdot,\eta))(\xi)\mathcal{F}_{D,B}(u)(\eta)d\mu_{k,\beta}(\eta)$$

is in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. Applying the inverse Dunkl-Bessel transform we get the result. ☐

Now we prove the fundamental result

Theorem 5.3. *Let $A(x,\triangle_{k,\beta})$ be the pseudo-differential-difference operator where the symbol $a(x,\xi):=A(x,-||\xi||^2)$ belongs to $S^{r,l}_{\exp,rad}$. Then for all u in $\mathcal{G}_*$ and all s in $\mathbb{R}$*

$$||A(x,\triangle_{k,\beta})u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} \leq C_s||u||_{W^{r+s,p}_{\mathcal{G}_*,k,\beta}}. \tag{5.3}$$

Proof. We consider the function

$$U_s(\xi)=e^{s||\xi||}\int_{\mathbb{R}^{d+1}_+}\tau_{-\eta}\big(\mathcal{F}_{D,B}(a)(\cdot,\eta)\big)(\xi)\mathcal{F}_{D,B}(u)(\eta)d\mu_{k,\beta}(\eta),\quad s\in\mathbb{R}.$$

Then invoking (5.2) and (2.19) we deduce that

$$|U_s(\xi)| \leq \int_{\mathbb{R}^{d+1}_+}\exp((r+s)\,||\eta||)|\mathcal{F}_{D,B}(u)(\eta)| \times \tau_{-\eta}\big(\exp(-(\tau-|s|)\,||y||)\big)(\xi)d\mu_{k,\beta}(\eta),\quad s\in\mathbb{R}. \tag{5.4}$$

The integral of (5.4) can be considered as a Dunkl-Bessel convolution product between $f(\xi)=\exp(-(\tau-|s|)\,||\xi||)$ and $g(\xi)=\exp((r+s)\,||\xi||)|\mathcal{F}_{D,B}(u)(\xi)|$. It is clear that f is radial and belongs to $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ for $\tau>|s|$, on the other hand since $\mathcal{F}_{D,B}(u)$ in $\mathcal{G}_*$ then g in $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$. Hence $f*_{D,B}g$ is in $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ and we have

$$||f*_{D,B}g||_{L^p_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq ||f||_{L^1_{k,\beta}(\mathbb{R}^{d+1}_+)}||g||_{L^p_{k,\beta}(\mathbb{R}^{d+1}_+)} = C_s||u||_{W^{r+s,p}_{\mathcal{G}_*,k,\beta}}.$$

Thus

$$||A(x,\triangle_{k,\beta})u||_{W^{s,p}_{\mathcal{G}_*,k,\beta}} = ||U_s||_{L^p_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq ||f*_{D,B}g||_{L^p_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq C_s||u||_{W^{r+s,p}_{\mathcal{G}_*,k,\beta}}.$$

This proves (5.3). □

5.2. **The Reproducing Kernels.**

Proposition 5.4. *For $s > 0$, the Hilbert space $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ admits the reproducing kernel:*

$$\Theta_{k,\beta}(x,y) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\Lambda(x,\xi)\Lambda(-y,\xi)e^{-2s||\xi||}d\mu_{k,\beta}(\xi),$$

that is:

i) *For every y in $\mathbb{R}^{d+1}_+$, the function $x \mapsto \Theta_{k,\beta}(x,y)$ belongs to $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$.*
ii) *For every f in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and y in $\mathbb{R}^{d+1}_+$, we have*

$$\langle f, \Theta_{k,\beta}(\cdot,y)\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} = f(y).$$

Proof. i) Let y be in $\mathbb{R}^{d+1}_+$. From (2.5), the function

$$\xi \mapsto \Lambda(-y,\xi)e^{-2s||\xi||}$$

belongs to $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)\bigcap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ for $s > 0$, then from Proposition 2.1 ii) there exists a function in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, which we denote by $\Theta_{k,\beta}(\cdot,y)$, such that

$$\mathcal{F}_{D,B}\big(\Theta_{k,\beta}(\cdot,y)\big)(\xi) = \Lambda(-y,\xi)e^{-2s||\xi||}. \tag{5.5}$$

Thus $\Theta_{k,\beta}(x,y)$ is given by

$$\Theta_{k,\beta}(x,y) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\Lambda(x,\xi)\Lambda(-y,\xi)e^{-2s||\xi||}d\mu_{k,\beta}(\xi).$$

ii) Let f be in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and y in $\mathbb{R}^{d+1}_+$. From (4.1),(5.5) and (2.15) we have

$$\langle f, \Theta_{k,\beta}(\cdot,y)\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\mathcal{F}_{D,B}(f)(\xi)\Lambda(y,\xi)d\mu_{k,\beta}(\xi) = f(y).$$

□

Proposition 5.5.

i) *Let f be in $\mathcal{G}_*$. Then $u(x,t) = \mathcal{H}_{k,\beta}(t)f(x)$ solves the problem*

$$\begin{cases} \left(\triangle_{k,\beta} - \frac{\partial}{\partial t}\right)u(x,t) = 0 \quad \text{on} \quad \mathbb{R}^{d+1}_+\times]0,\infty[\\ u(\cdot,0) = f. \end{cases}$$

ii) *The integral transform $\mathcal{H}_{k,\beta}(t)$, $t > 0$, is a bounded linear operator from $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$, s in $\mathbb{R}$, into $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, and we have*

$$\|\mathcal{H}_{k,\beta}(t)f\|_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} \le e^{\frac{s^2}{4t}}\|f\|_{W^{s,2}_{\mathcal{G}_*,k,\beta}}.$$

Proof. i) This assertion follows from Definition 3.4 and Proposition 3.4 iii).

ii) Let f be in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$. Using Proposition 2.1 ii) we have

$$\|\mathcal{H}_{k,\beta}(t)f\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} = m_{k,\beta}\|\mathcal{F}_{D,B}(\mathcal{H}_{k,\beta}(t)f)\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}.$$

Invoking the relation ships (2.21) and (3.2) we can write

$$\|\mathcal{H}_{k,\beta}(t)f\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} e^{-2t||\xi||^2}|\mathcal{F}_{D,B}(f)(\xi)|^2 d\mu_{k,\beta}(\xi).$$

Therefore

$$\|\mathcal{H}_{k,\beta}(t)f\|_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} \leq e^{\frac{s^2}{4t}}\|f\|_{W^{s,2}_{\mathcal{G}_*,k,\beta}}.$$

□

Definition 5.3. Let $r > 0$, $t \geq 0$ and s in $\mathbb{R}$. We define the Hilbert space $H^{r,s}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ as the subspace of $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ with the inner product:

$$\langle f, g\rangle_{H^{r,s}_{\mathcal{G}_*,k,\beta}} = r\langle f, g\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} + \langle \mathcal{H}_{k,\beta}(t)f, \mathcal{H}_{k,\beta}(t)g\rangle_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}, \quad f, g \in W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{H^{r,s}_{\mathcal{G}_*,k,\beta}} := r\|f\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} + \|\mathcal{H}_{k,\beta}(t)f\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}.$$

Proposition 5.6. *For $s \in \mathbb{R}$, the Hilbert space $H^{r,s}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ admits the following reproducing kernel:*

$$P_r(x, y) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} \frac{\Lambda(x, \xi)\Lambda(-y, \xi)d\mu_{k,\beta}(\xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}.$$

Proof. i) Let y be in $\mathbb{R}^{d+1}_+$. In the same way as in the proof of Proposition 5.4 i), we can prove that the function $x \mapsto P_r(x, y)$ belongs to $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and we have

$$\mathcal{F}_{D,B}\big(P_r(\cdot, y)\big)(\xi) = \frac{\Lambda(-y, \xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}. \tag{5.6}$$

On the other hand we have

$$\mathcal{F}_{D,B}\big(\mathcal{H}_{k,\beta}(t)(P_r(\cdot, y)\big)(\xi) = \exp(-t||\xi||^2)\mathcal{F}_{D,B}\big(P_r(\cdot, y)\big)(\xi), \quad \text{for all} \quad \xi \in \mathbb{R}^{d+1}_+. \tag{5.7}$$

Hence from Proposition 2.1 ii), we obtain

$$\begin{aligned}\|\mathcal{H}_{k,\beta}(t)(P_r(\cdot, y)\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} &= m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} e^{-2t||\xi||^2}\left|\mathcal{F}_{D,B}\big(P_r(\cdot, y)\big)(\xi)\right|^2 d\mu_{k,\beta}(\xi)\\ &\leq \frac{C}{r^2}\int_{\mathbb{R}^{d+1}_+} \frac{e^{-2t||\xi||^2}}{e^{4s||\xi||}}d\mu_{k,\beta}(\xi) < \infty.\end{aligned}$$

Therefore we conclude that $\|P_r(\cdot, y)\|^2_{H^{r,s}_{\mathcal{G}_*,k,\beta}} < \infty$.

ii) Let f be in $H^{r,s}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and y in $\mathbb{R}^{d+1}_+$. Then

$$\langle f, P_r(\cdot, y)\rangle_{H^{r,s}_{\mathcal{G}_*,k,\beta}} = rI_1 + I_2, \tag{5.8}$$

where

$$\begin{aligned} I_1 &= \langle f, P_r(\cdot, y)\rangle_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \quad \text{and}\\ I_2 &= \langle \mathcal{H}_{k,\beta}(t)f, \mathcal{H}_{k,\beta}(t)(P_r(\cdot, y))\rangle_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

From (5.6) and (4.1), we have

$$I_1 = m_{k,\beta} \int_{\mathbb{R}_+^{d+1}} \frac{e^{2s||\xi||}\mathcal{F}_{D,B}(f)(\xi)\Lambda(y,\xi)d\mu_{k,\beta}(\xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}.$$

From (5.7) and (5.6) and Proposition 2.1 ii) we have

$$I_2 = m_{k,\beta} \int_{\mathbb{R}_+^{d+1}} \frac{e^{-2t||\xi||^2}\mathcal{F}_{D,B}(f)(\xi)\Lambda(y,\xi)d\mu_{k,\beta}(\xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}.$$

The relations (5.8) and (2.15) imply that

$$\langle f, P_r(\cdot,y)\rangle_{H^{r,s}_{\mathcal{G}_*,k,\beta}} = f(y).$$

□

5.3. **Extremal Function for the Generalized Heat Semigroup Transform.** In this subsection, we prove for a given function g in $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ that the infimum of

$$\left\{r\|f\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} + \|g - \mathcal{H}_{k,\beta}(t)f\|^2_{L^2_{k,\beta}(\mathbb{R}_+^{d+1})},\ f \in W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}_+^{d+1})\right\}$$

is attained at some unique function denoted by $f^*_{r,g}$, called the extremal function. We start with the following fundamental theorem (cf. [11, 18]).

Theorem 5.7. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E and H a Hilbert space. Let $L : H_K \to H$ be a bounded linear operator on H_K into H. For $r > 0$, introduce the inner product in H_K and call it H_{K_r} as*

$$\langle f_1, f_2\rangle_{H_{K_r}} = r\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_H.$$

Then

i) *H_{K_r} is the Hilbert space with the reproducing kernel $K_r(p,q)$ on E which satisfies the equation*

$$K(\cdot,q) = (rI + L^*L)K_r(\cdot,q),$$

where L^* is the adjoint operator of $L : H_K \to H$.

ii) *For any $r > 0$ and for any g in H, the infimum*

$$\inf_{f\in H_K} \left\{r\|f\|^2_{H_K} + \|Lf - g\|^2_H\right\}$$

is attained by a unique function $f^*_{r,g}$ in H_K and this extremal function is given by

$$f^*_{r,g}(p) = \langle g, LK_r(\cdot,p)\rangle_H. \tag{5.9}$$

We can now state the main result of this subsection.

Theorem 5.8. *Let $s \in \mathbb{R}$. For any g in $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ and for any $r > 0$, the infimum*

$$\inf_{f\in W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}_+^{d+1})} \left\{r\|f\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} + \|g - \mathcal{H}_{k,\beta}(t)f\|^2_{L^2_{k,\beta}(\mathbb{R}_+^{d+1})}\right\} \tag{5.10}$$

is attained by a unique function $f^*_{r,g}$ *given by*

$$f^*_{r,g}(x) = \int_{\mathbb{R}^{d+1}_+} g(y)Q_r(x,y)d\mu_{k,\beta}(y), \tag{5.11}$$

where

$$\begin{aligned} Q_r(x,y) &= Q_{r,s}(x,y) \\ &= m_{k,\beta}\int_{\mathbb{R}^{d+1}_+} \frac{e^{-t\|\xi\|^2}\Lambda(x,\xi)\Lambda(-y,\xi)}{re^{2s\|\xi\|}+e^{-2t\|\xi\|^2}}d\mu_{k,\beta}(\xi). \end{aligned} \tag{5.12}$$

Proof. By Proposition 5.6 and Theorem 5.7 ii), the infimum given by (5.10) is attained by a unique function $f^*_{r,g}$, and from (5.9) the extremal function $f^*_{r,g}$ is represented by

$$f^*_{r,g}(y) = \langle g, \mathcal{H}_{k,\beta}(t)(P_r(\cdot,y))\rangle_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}, \quad y\in\mathbb{R}^{d+1}_+,$$

where P_r is the kernel given by Proposition 5.6. On the other hand we have

$$\mathcal{H}_{k,\beta}(t)f(x) = m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\exp(-t\|\xi\|^2)\mathcal{F}_{D,B}(f)(\xi)\Lambda(x,\xi)d\mu_{k,\beta}(\xi), \quad \text{for all} \quad x\in\mathbb{R}^{d+1}_+.$$

Hence by (5.6), we obtain

$$\begin{aligned} \mathcal{H}_{k,\beta}(t)\big(P_r(\cdot,y)\big)(x) &= m_{k,\beta}\int_{\mathbb{R}^{d+1}_+}\frac{\exp(-t\|\xi\|^2)\Lambda(x,\xi)\Lambda(-y,\xi)}{re^{2s\|\xi\|}+e^{-2t\|\xi\|^2}}d\mu_{k,\beta}(\xi) \\ &= Q_r(x,y). \end{aligned}$$

This gives (5.12). □

Corollary 5.9. *Let* $s\in\mathbb{R}$, $\delta>0$ *and* g, g_δ *in* $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ *such that*

$$\|g-g_\delta\|_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} \le \delta.$$

Then

$$\|f^*_{r,g}-f^*_{r,g_\delta}\|_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \le \frac{\delta}{2\sqrt{r}}.$$

Proof. From (5.12) and Fubini's theorem we have

$$\mathcal{F}_{D,B}(f^*_{r,g})(\xi) = \frac{e^{-t\|\xi\|^2}\mathcal{F}_{D,B}(g)(\xi)}{re^{2s\|\xi\|}+e^{-2t\|\xi\|^2}}. \tag{5.13}$$

Hence

$$\mathcal{F}_{D,B}(f^*_{r,g}-f^*_{r,g_\delta})(\xi) = \frac{e^{-t\|\xi\|^2}\mathcal{F}_{D,B}(g-g_\delta)(\xi)}{re^{2s\|\xi\|}+e^{-2t\|\xi\|^2}}.$$

Using the inequality $(x+y)^2\ge 4xy$, we obtain

$$e^{2s\|\xi\|}\left|\mathcal{F}_{D,B}(f^*_{r,g}-f^*_{r,g_\delta})(\xi)\right|^2 \le \frac{1}{4r}|\mathcal{F}_{D,B}(g-g_\delta)(\xi)|^2.$$

This and Proposition 2.1 ii) give

$$\begin{aligned} \|f^*_{r,g}-f^*_{r,g_\delta}\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} &\le \frac{m_{k,\beta}}{4r}\|\mathcal{F}_{D,B}(g-g_\delta)\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} \\ &\le \frac{1}{4r}\|g-g_\delta\|^2_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)}, \end{aligned}$$

from which we obtain the desired result. □

Corollary 5.10. *Let $s \in \mathbb{R}$. If f is in $W^{s,2}_{\mathcal{G}_*,k,\beta}(\mathbb{R}^{d+1}_+)$ and $g = \mathcal{H}_{k,\beta}(t)f$. Then*

$$\|f^*_{r,g} - f\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} \to 0 \quad \textit{as} \;\; r \to 0.$$

Proof. From (3.2), (5.13) we have

$$\mathcal{F}_{D,B}(f)(\xi) = \exp(t||\xi||^2)\mathcal{F}_{D,B}(g)(\xi)$$

and

$$\mathcal{F}_{D,B}(f^*_{r,g})(\xi) = \frac{e^{-t||\xi||^2}\mathcal{F}_{D,B}(g)(g)(\xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}.$$

Hence

$$\mathcal{F}_{D,B}(f^*_{r,g} - f)(\xi) = \frac{-re^{2s||\xi||}\mathcal{F}_{D,B}(f)(\xi)}{re^{2s||\xi||} + e^{-2t||\xi||^2}}.$$

Then we obtain

$$\|f^*_{r,g} - f\|^2_{W^{s,2}_{\mathcal{G}_*,k,\beta}} = \int_{\mathbb{R}^{d+1}_+} h_{r,t,s}(\xi)|\mathcal{F}_{D,B}(f)(\xi)|^2 d\mu_{k,\beta}(\xi),$$

with

$$h_{r,t,s}(\xi) = \frac{r^2 e^{6s||\xi||}}{(re^{2s||\xi||} + e^{-2t||\xi||^2})^2}.$$

Since

$$\lim_{r\to 0} h_{r,t,s}(\xi) = 0$$

and

$$|h_{r,t,s}(\xi)| \le e^{2s||\xi||},$$

we obtain the result from the dominated convergence theorem. □

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