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ON A NEW TYPE OF MEYER-KONIG AND ZELLER OPERATORS

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ABSTRACT. In this present paper, we introduce a new and simple integral modification of the Meyer-Konig and Zeller Bezier type operators and study the rate of convergence for functions of bounded variation. Our result improves and corrects the results of Guo (*J. Approx. Theory*, **56** (1989), 245–255), Zeng (*Comput. Math. Appl.*, **39** (2000), 1–13; *J. Math. Anal. Appl.*, **219** (1998), 364–376), etc.

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1. INTRODUCTION

For a function defined on $[0,1]$, the Meyer-Konig and Zeller operators P_n, $n \in \mathbb{N}$ [6] are defined by

$$P_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right), \; x \in [0,1], \tag{1.1}$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n .$$

The rates of convergence of some integral modifications of the operators (1.1) were discussed in [1] – [5] and [7]. Recently, Zeng [8] generalized the operators (1.1) and its integral modification and estimated the rate of convergence for functions of bounded variation. We introduce a new generalization of the Meyer-Konig-Zeller operators for functions defined on $[0,1]$ as

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^1 b_{n,k}(t) f(t)\, dt, \; x \in [0,1], \tag{1.2}$$

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where $\alpha \geq 1$,

$$Q_{n,k}^{(\alpha)}(x) = \left(\sum_{j=k}^{\infty} m_{n,j}(x)\right)^{\alpha} - \left(\sum_{j=k+1}^{\infty} m_{n,j}(x)\right)^{\alpha}$$

and

$$b_{n,k}(t) = \frac{(n+k)!}{k!\,(n-1)!} t^k (1-t)^{n-1}.$$

For further properties of $Q_{n,k}^{(\alpha)}(x)$, we refer readers to [8]. Actually, the main purpose in introducing these operators (1.2) is that some approximation formulae for $B_{n,\alpha}$, $n \in \mathbb{N}$ become simpler than the corresponding results for the other modifications considered in [8]. For $\alpha = 1$, the operators (1.2) reduce to

$$B_{n,1}(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)\, dt.$$

In the present paper, we study the behaviour of $B_{n,\alpha}(f,x)$ for functions of bounded variation and give an estimate on the rate of convergence for these new integrated Meyer-Konig-Zeller-Bezier operators. In the last section, we give the correct estimates for some other generalized Meyer-Konig-Zeller type operators considered in [2], [7] and [8].

2. AUXILIARY RESULTS

In this section we give certain results, which are necessary to prove the main result.

Lemma 2.1. [7]. *For all* $k, n \in \mathbb{N}$, $x \in (0,1]$, *we have*

$$m_{n,k}(x) < \frac{1}{\sqrt{2enx}},$$

where the constant $\frac{1}{\sqrt{2e}}$ *is the best possible.*

Lemma 2.2. *For* $r \in \mathbb{N}^0$ *(the set of non negative integers), if we define*

$$B_{n,1}\left((t-x)^r, x\right) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) (t-x)^r\, dt,$$

then

$$B_{n,1}\left((t-x)^2, x\right) \leq \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}.$$

Proof. Put $e_r(x) = x^r$, $r = 0, 1, 2, \ldots$. The moments of the operators $B_{n,1}f$ are given by

$$B_{n,1}(t^r, x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n+k)!}{k!\,(n-1)!} \mathbf{B}(k+r+1, n),$$

where $\mathbf{B}$ is the Beta function. Obviously $B_{n,1}e^0 = e^0 = 1$.

$$\begin{aligned} B_{n,1}(t,x) &= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \frac{(k+1)}{(n+k+1)} \\ &\geq (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-2}{k-1} x^k \frac{(k+1)}{k} \cdot \frac{(n+k-1)}{(n+k+1)} \\ &\geq (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{k+1} \left[1 - \frac{2}{n+1}\right] \\ &= \left[1 - \frac{2}{n+1}\right] x. \end{aligned}$$

Next,

$$\begin{aligned} B_{n,1}\left(t^2,x\right) &= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \frac{k(k-1)+4k+2}{(n+k+1)(n+k+2)} \\ &\leq \frac{(1-x)^n}{(n-1)!} \sum_{k=2}^{\infty} \frac{(n+k-3)!}{(k-2)!} x^k + 4 \sum_{k=1}^{\infty} \frac{(n+k-3)!}{(k-1)!} x^k \\ &\quad + 2 \sum_{k=0}^{\infty} \frac{(n+k-3)!}{k!} x^k \\ &\leq (1-x)^n \left\{ \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{k+2} + \frac{4}{(n-1)} \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{k+1} \right. \\ &\quad \left. + \frac{2}{(n-1)(n-2)} \binom{n+k-1}{k} x^k \right\} \\ &= x^2 + \frac{4x(1-x)}{(n-1)} + \frac{2(1-x)^2}{(n-1)(n-2)}. \end{aligned}$$

Combining these, we get

$$\begin{aligned} B_{n,1}\left((t-x)^2,x\right) &= B_{n,1}\left(t^2,x\right) - 2xB_{n,1}(t,x) + x^2 \\ &\leq \frac{4x}{(n-1)} + \frac{2(1-x)^2}{(n-1)(n-2)}. \end{aligned}$$

In particular, given any $\lambda > 4$ and any $x \in (0,1)$, there is an integer $N(\lambda,x)$ such that for all $n \geq N(\lambda,x)$

$$B_{n,1}\left((t-x)^2,x\right) \leq \frac{\lambda x}{n}.$$

□

Lemma 2.3. *For all $x \in (0,1]$ and $k \in \mathbb{N}$, there holds*

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha m_{n,k}(x) < \frac{\alpha}{\sqrt{2enx}}.$$

Proof. It is easy to verify that $|a^\alpha - b^\alpha| \leq \alpha |a-b|$, $(0 \leq a,\ b \leq 1,\ \alpha \geq 1)$. Then by Lemma 2.1, we obtain

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha m_{n,k}(x) < \frac{\alpha}{\sqrt{2enx}}.$$

□

Lemma 2.4. *Let* $K_{n,\alpha}(x,t)=\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\,b_{n,k}(t)$, *and* $\lambda>4,\, n\geq N(\lambda,x)$ *then*

$$\lambda_{n,\alpha}(x,y)=\int_0^y K_{n,\alpha}(x,t)\,dt\leq\frac{\alpha\lambda x}{n(x-y)^2},\qquad 0\leq y<x, \tag{2.1}$$

$$1-\lambda_{n,\alpha}(x,y)=\int_z^1 K_{n,\alpha}(x,t)\,dt\leq\frac{\alpha\lambda x}{n(z-x)^2},\qquad x<z\leq 1. \tag{2.2}$$

Proof. We first prove (2.1), as follows

$$\begin{aligned}\int_0^y K_{n,\alpha}(x,t)\,dt &\leq \int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}dt\\ &\leq \frac{1}{(x-y)^2}B_{n,\alpha}\left((t-x)^2,x\right)\\ &\leq \frac{\alpha B_{n,1}\left((t-x)^2,x\right)}{(x-y)^2}\\ &\leq \frac{\alpha\lambda x}{n(x-y)^2},\end{aligned}$$

by Lemma 2.2. The proof of (2.2) is similar. □

Lemma 2.5. [8, p. 5]. *For* $x\in(0,1)$*, we have*

$$\left|\sum_{\frac{nx}{1-x}<k} m_{n,k}(x)-\frac{1}{2}\right|\leq\frac{5}{2\sqrt{nx}}.$$

3. MAIN RESULT

In this section we prove the following main theorem

Theorem 3.1. *Let* f *be a function of bounded variation on* $[0,1]$, $\alpha\geq 1$. *Then for every* $x\in(0,1)$ *and* $\lambda>4$ *and* $n\geq\max\{N(\lambda,x),3\}$*, we have*

$$\begin{aligned}&\left|B_{n,\alpha}(f,x)-\left[\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right]\right|\\ &\qquad\leq\frac{1}{2}\left[\frac{\alpha^2+\alpha-2}{\alpha+1}+\frac{\alpha}{\sqrt{2enx}}\right]|f(x+)-f(x-)|+\frac{(2\lambda\alpha+x)}{nx}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x),\end{aligned}$$

where

$$g_x(t)=\begin{cases} f(t)-f(x-), & 0\leq t<x;\\ 0, & t=x;\\ f(t)-f(x+), & x<t\leq 1\end{cases}$$

and $\bigvee_a^b(g_x)$ *is the total variation of* g_x *on* $[a,b]$.

Proof. Clearly

$$\left|B_{n,\alpha}(f,x)-\left[\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right]\right| \leq \left|B_{n,\alpha}(\operatorname{sgn}(t-x),x)+\left(\frac{\alpha-1}{\alpha+1}\right)\right|\cdot\frac{|f(x+)-f(x-)|}{2}+|B_{n,\alpha}(g_x,x)|. \tag{3.1}$$

First, we have

$$\begin{aligned}B_{n,\alpha}(\operatorname{sgn}(t-x),x) &= \sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\int_x^1 b_{n,k}(t)\,dt-\int_0^x b_{n,k}(t)\,dt\right)\\ &= \sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\int_0^1 b_{n,k}(t)\,dt-2\int_0^x b_{n,k}(t)\,dt\right)\\ &= 1-2\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_0^x b_{n,k}(t)\,dt.\end{aligned}$$

Using Lemma 2.1, Lemma 2.3 and the fact that $\sum_{j=0}^{k}m_{n,j}(x)=\int_x^1 b_{n,k}(t)\,dt$, we have

$$\begin{aligned}B_{n,\alpha}(\operatorname{sgn}(t-x),x) &= 1-2\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(1-\sum_{j=0}^{k}m_{n,j}(x)\right)\\ &= -1+2\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\sum_{j=0}^{k}m_{n,j}(x)\right)\\ &\leq -1+2\alpha\sum_{k=0}^{\infty}m_{n,k}(x)\sum_{j=0}^{k}m_{n,j}(x)\\ &= -1+\alpha+\alpha\left[\sum_{k=0}^{\infty}m_{n,k}(x)\sum_{j=0}^{k}m_{n,j}(x)-\sum_{k=0}^{\infty}(m_{n,k}(x))^2\right]\\ &\leq \alpha-1+\alpha m_{n,k}(x)\sum_{k=0}^{\infty}m_{n,k}(x)\\ &\leq \alpha-1+\frac{\alpha}{\sqrt{2enx}}.\end{aligned}$$

Thus

$$\left|B_{n,\alpha}(\operatorname{sgn}(t-x),x)+\left(\frac{\alpha-1}{\alpha+1}\right)\right|\leq\frac{\alpha^2+\alpha-2}{\alpha+1}+\frac{\alpha}{\sqrt{2enx}}. \tag{3.2}$$

Next we estimate $B_{n,\alpha}(g_x,x)$. By the Lebesgue-Stieltjes integral representation, we have

$$\begin{aligned}B_{n,\alpha}(g_x,x) &= \int_0^1 K_{n,\alpha}(x,t)\,g_x(t)\,dt\\ &= \left(\int_{I_1}+\int_{I_2}+\int_{I_3}\right)K_{n,\alpha}(x,t)\,g_x(t)\,dt\\ &= E_1+E_2+E_3, \qquad \text{say,}\end{aligned}$$

where

$$I_1=\left[0,x-\frac{x}{\sqrt{n}}\right],\quad I_2=\left[x-\frac{x}{\sqrt{n}},x+\frac{1-x}{\sqrt{n}}\right]\ \text{ and }\ I_3=\left[x+\frac{1-x}{\sqrt{n}},1\right].$$

We first estimate E_1. Writing $y = x - \frac{x}{\sqrt{n}}$ and using Lebesgue-Stieltjes integration by parts with $\lambda_{n,\alpha}(x,t) = \int_0^t K_{n,\alpha}(x,u)\,du$, we have

$$E_1 = \int_0^y g_x(t)\,d_t(\lambda_{n,\alpha}(x,t)) = g_x(y+)\,\lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t)\,d_t(g_x(t)).$$

Since $|g_x(y+)| \leq V_{y+}^x(g_x)$, it follows that

$$|E_1| \leq \bigvee_{y+}^{x}(g_x)\,\lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t)\,d_t\left(-\bigvee_t^x(g_x)\right).$$

By using (2.1) of Lemma 2.4, we get

$$|E_1| \leq \bigvee_{y+}^{x}(g_x)\frac{\alpha\lambda x}{n(x-y)^2} + \frac{\alpha\lambda x}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_x)\right).$$

Integrating the last term by parts we have after simple computation

$$|E_1| \leq \frac{\alpha\lambda x}{n}\left[\frac{\bigvee_0^x(g_x)}{x^2} + 2\int_0^y \frac{\bigvee_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - \frac{x}{\sqrt{u}}$, we obtain

$$|E_1| \leq \frac{\alpha\lambda}{nx}\left[\bigvee_0^x(g_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x}\right] \leq \frac{2\lambda\alpha}{nx}\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x). \tag{3.3}$$

Using the similar method of Lemma 2.4 and equation (2.2), we get

$$|E_3| \leq \frac{2\alpha\lambda}{nx}\sum_{k=1}^n \bigvee_{x}^{x+\frac{1-x}{\sqrt{k}}}(g_x). \tag{3.4}$$

Finally we estimate E_2.

For $t \in \left[x - \frac{x}{\sqrt{n}}, x + \frac{1-x}{\sqrt{n}}\right]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leq \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x),$$

and therefore

$$|E_2| \leq \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x)\int_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}} d_t(\lambda_{n,\alpha}(x,t)).$$

Since $\int_a^b d_t\lambda_n(x,t)\,dt \leq 1$, for $(a,b) \subset [0,1]$, therefore

$$|E_2| \leq \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x) \leq \frac{1}{n}\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x). \tag{3.5}$$

Finally collecting the estimates (3.2) – (3.5), we obtain

$$|B_{n,\alpha}(g_x,x)| \leq \frac{(2\lambda\alpha + x)}{nx}\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x). \tag{3.6}$$

Combining (3.1), (3.2) and (3.6), our theorem follows. □

4. SOME EXAMPLES

Recently Zeng [8] introduced some other generalized Meyer-Konig and Zeller operators as

$$\widehat{M}_{n,\alpha}(f,x)=\sum_{k=0}^{\infty}\left(\frac{Q_{n,k}^{(\alpha)}(x)}{\int_{I_k}1dt}\right)\int_{I_k}f(t)\,dt=\int_0^1 f(t)\,K_{n,\alpha,2}(x,t)\,dt, \tag{4.1}$$

where $\alpha\geq 1$, $K_{n,\alpha,2}(x,t)=\sum_{k=0}^{\infty}\frac{Q_{n,k}^{(\alpha)}\chi_k(t)}{\int_I\chi_k(u)du}$ and χ_k is the characteristic function of the interval $I_k=\left[\frac{k}{n+k},\frac{k+1}{n+k+1}\right]$ with respect to $I=[0,1]$ and $Q_{n,k}^{(\alpha)}(x)$ is as defined by (1.1). In particular, for $\alpha=1$, the operators (4.1) reduce to the operators $\widehat{M}_{n,1}(f,x)$ studied by Guo [2], as

$$\widehat{M}_{n,1}(f,x)=\sum_{k=0}^{\infty}\widehat{m}_{n,k}(x)\int_{I_k}f(t)\,dt, \tag{4.2}$$

where

$$\widehat{m}_{n,k}(x)=(n+1)\begin{pmatrix}n+k+1\\k\end{pmatrix}x^k(1-x)^n.$$

We have noticed that Theorem 2 in [8] and Theorem 4.2 in [7] are not correct, there are some misprints. This motivated us to correct these estimates and in this section we have been able to correct and achieve improved estimates over the results of Zeng ([7, 8]), Guo [2] and Love et al. [5].

The misprinted estimate obtained by Zeng [8] is as follows:

Theorem 4.1. *Let f be a function of bounded variation on $[0,1]$. Then for every $x\in(0,1)$ and n sufficiently large, we have*

$$\left|\widehat{M}_{n,\alpha}(f,x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right|$$
$$\leq\frac{5\alpha}{\sqrt{nx}+1}\left|f(x+)-f(x-)\right|+\frac{5\alpha}{nx+1}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x),$$

where $g_x(t)$ and $\bigvee_a^b(g_x)$ are as defined by Theorem 3.1.

Remark 4.2. It is remarked that there are misprints in Lemma 6 and Lemma 8 of Zeng [8]. Actually the author [8] has not verified his Lemmas 6 and 8 and he had taken these results directly from the paper of Guo [2] (see Lemmas 5 and 6 of [2]). As the Lemmas 5 and 6 of [2] are not correct. Although these mistakes were pointed out earlier by Love et al. [5] in 1994. Zeng also has not gone through the paper of Love et al. [5] and in another paper he has obtained the following misprinted exact bound for the operators (4.2):

Theorem 4.3. *Let f be a function of bounded variation on $[0,1]$. Then for every $x\in(0,1)$ and n sufficiently large, we have*

$$\left|\widehat{M}_{n,1}(f,x)-\frac{1}{2}\left[f(x+)-f(x-)\right]\right|$$
$$\leq\left(16+\frac{1}{\sqrt{2e}}\right)\frac{1}{x^{\frac{3}{2}}\sqrt{n}}\left|f(x+)-f(x-)\right|+\frac{5}{nx}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x),$$

where $g_x(t)$ and $\bigvee_a^b(g_x)$ are as defined by Theorem 3.1.

Remark 4.4. It may be remarked here that the factor $\frac{5}{nx}$ in Theorem 4.3 and $\frac{5\alpha}{nx+1}$ in Theorem 4.1 are not correct as the second moment estimate of Guo [2] was not obtained correctly.

We now give the correct and improved statement of Theorem 4.1 with an outline of the proof. For $\alpha = 1$ our result reduces to the corresponding improved version of Theorem 4.3. Before giving the theorem we shall give a lemma which is necessary in the proof of our Theorem 4.6.

Lemma 4.5. *Let $n > 2$, $\omega_n = \frac{4}{(n-1)\delta^2}\left(1+\frac{1-\delta}{12(n-2)\delta}\right)$ where $0 < \delta \le x \le 1-\delta$, then*

(i) *for $0 < y < x$ there holds*

$$\int_0^y K_{n,\alpha,2}(x,t)\,dt \le \frac{\alpha\omega_n x^3(1-x)}{(x-y)^2}; \tag{4.3}$$

(ii) *for $x < y < 1$ there holds*

$$\int_z^1 K_{n,\alpha,2}(x,t)\,dt \le \frac{\alpha\omega_n x(1-x)^3}{(z-x)^2}. \tag{4.4}$$

Proof. Following [5, Lemma 7] for $x \in (0,1)$, $n > 2$, we have

$$\widehat{M}_{n,1}\left((t-x)^2, x\right) \le \frac{4x(1-x)}{(n-1)} + \frac{1}{3}\cdot\frac{(1-x)^2}{(n-1)(n-2)}. \tag{4.5}$$

Using (4.5), we have

$$\begin{aligned}
\int_0^y K_{n,\alpha,2}(x,t)\,dt &\le \int_0^y K_{n,\alpha,2}(x,t)\frac{(x-t)^2}{(x-y)^2}dt \\
&\le \frac{\alpha}{(x-y)^2}\widehat{M}_{n,1}\left((t-x)^2, x\right) \\
&\le \frac{\alpha}{(x-y)^2}\left[\frac{4x(1-x)}{(n-1)} + \frac{(1-x)^2}{3(n-1)(n-2)}\right] \\
&\le \frac{4\alpha x^3(1-x)}{\delta^2(n-1)(x-y)^2}\left[1+\frac{\frac{1}{\delta}-1}{12(n-2)}\right] \\
&= \frac{\alpha\omega_n x^3(1-x)}{(x-y)^2}.
\end{aligned}$$

This completes the proof of (4.3). The proof of (4.4) is similar. □

Theorem 4.6. *Let f be a function of bounded variation on $[0,1]$. Then for every $x \in (0,1)$, $n > 2$ and $0 < \delta \le x \le 1-x$, we have*

$$\begin{aligned}
&\left|\widehat{M}_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \\
&\qquad \le \left(\frac{5}{2}+\frac{1}{\sqrt{2e}}\right)\frac{\alpha}{\sqrt{nx}}\left|f(x+)-f(x-)\right| \\
&\qquad\qquad + \left(2\alpha\omega_n x(1-x) + \frac{1}{n-1}\right)\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x),
\end{aligned}$$

where $g_x(t)$ and $\bigvee_a^b(g_x)$ are as defined by Theorem 3.1 and ω_n is as given by Lemma 4.5.

Proof. First

$$\begin{aligned}\left|\widehat{M}_{n,\alpha}(f,x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \\ \le \left|\widehat{M}_{n,\alpha}(g_x,x)\right|+\left|\frac{f(x+)-f(x-)}{2^\alpha}\widehat{M}_{n,\alpha}(\operatorname{sgn}(t-x),x)\right. \\ \left.+\left[f(x+)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right]\widehat{M}_{n,\alpha}(\delta_x,x)\right|.\end{aligned}$$

Following [8], we have $\widehat{M}_{n,\alpha}(\delta_x,x)=0$ and

$$\widehat{M}_{n,\alpha}(\operatorname{sgn}(t-x),x)\le\alpha 2^\alpha\left|\sum_{\frac{nx}{1-x}<k}m_{n,k}(x)-\frac{1}{2}\right|+2^\alpha Q_{n,k'}^{(\alpha)}(x)$$

where $x\in\left[\frac{k'}{n+k'},\frac{k'+1}{n+k'+1}\right)$.

Using Lemma 2.3 and Lemma 2.5, we have

$$\left|\widehat{M}_{n,\alpha}(\operatorname{sgn}(t-x),x)\right|\le\left(\frac{5}{2}+\frac{1}{\sqrt{2e}}\right)\frac{\alpha 2^\alpha}{\sqrt{nx}}.$$

Next, we estimate $\widehat{M}_{n,\alpha}(g_x,x)$, as follows

$$\begin{aligned}\widehat{M}_{n,\alpha}(g_x,x)&=\int_0^1 K_{n,\alpha,2}(x,t)\,g_x(t)\,dt\\ &=\left(\int_0^{x-\frac{x}{\sqrt{n}}}+\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}+\int_{x+\frac{1-x}{\sqrt{n}}}^1\right)K_{n,\alpha,2}(x,t)\,g_x(t)dt\\ &=R_1+R_2+R_3,\ \text{say.}\end{aligned}$$

The evaluation of R_1,R_2,R_3 are similar to work in [8]. We have

$$|R_2|\le\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}(g_x)\le\frac{1}{n-1}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}(g_x).$$

Next suppose $y=x-\frac{x}{\sqrt{n}}$. Using integration by parts with $\mu_{n,\alpha}(x,t)=\int_0^t K_{n,\alpha,2}(x,u)\,du$, we have

$$\begin{aligned}R_1&=\int_0^y g_x(t)\,d_t(\mu_{n,\alpha}(x,t))\\ &=g_x(y+)\,\mu_{n,\alpha}(x,y)-\int_0^y\mu_{n,\alpha}(x,t)\,d_t(g_x(t))\\ &\le\bigvee_{y+}^{x}(g_x)\,\mu_{n,\alpha}(x,y)+\int_0^y\mu_{n,\alpha}(x,t)\,d_t\left(-\bigvee_t^x(g_x)\right).\end{aligned}$$

By (4.3) of Lemma 4.5, we have

$$|R_1|\le\bigvee_{y+}^{x}(g_x)\frac{\alpha\omega_n x^3(1-x)}{(x-y)^2}+\alpha\omega_n x^3(1-x)\int_0^y\frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_x)\right).$$

Integrating the last term by parts we get

$$|R_1| \leq \alpha\omega_n x^3 (1-x) \left[\frac{\bigvee_0^x (g_x)}{x^2} + 2\int_0^y \frac{-\bigvee_t^x (g_x)}{(x-t)^3} dt \right].$$

Now replacing the variable y in the last integral by $x - \frac{x}{\sqrt{u}}$, we have

$$\begin{aligned} |R_1| &\leq \frac{1}{x^2} \left[\bigvee_0^x (g_x) + \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x} \right] \alpha\omega_n x^3 (1-x) \\ &\leq 2\alpha\omega_n x (1-x) \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x} (g_x). \end{aligned}$$

Finally using the similar methods, we have

$$|R_3| \leq 2\alpha\omega_n x (1-x) \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x} (g_x).$$

Combining the estimates of R_1, R_2, R_3, our theorem follows. □

Remark 4.7. In particular, for $\alpha = 1$, by Theorem 4.6 it may be remarked that the main theorem of Love et al. [5] i.e. $\left(\frac{5}{x\sqrt{nx}}\right) |f(x+) - f(x-)|$ can be improved to

$$\left\{ \left(\frac{5}{2} + \frac{1}{\sqrt{2e}} \right) \Big/ \sqrt{nx} \right\} |f(x+) - f(x-)|.$$

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