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NEW PERTURBED ITERATIONS FOR A GENERALIZED CLASS OF STRONGLY NONLINEAR OPERATOR INCLUSION PROBLEMS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to introduce and study a new kind of generalized strongly nonlinear operator inclusion problems involving generalized m-accretive mapping in Banach spaces. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang, we also prove the existence theorem of the solution for this kind of operator inclusion problems and construct a new class of perturbed iterative algorithm with mixed errors for solving this kind of generalized strongly nonlinear operator inclusion problems in Banach spaces. Further, we discuss the convergence and stability of the iterative sequence generated by the perturbed algorithm. Our results improve and generalize the corresponding results of [3, 6, 11, 12].

Key words and phrases: Generalized m-accretive mapping; Generalized strongly nonlinear operator inclusion problems; Perturbed iterative algorithm with errors; Existence; Convergence and stability.

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1. INTRODUCTION

Let X be a real Banach space and $T : X \to 2^X$ is a multi-valued operator, where 2^X denotes the family of all the nonempty subsets of X. The following operator inclusion problem of finding $x \in X$ such that

$$0 \in T(u) \tag{1.1}$$

has been studied extensively because of its role in the modelization of unilateral problems, nonlinear dissipative systems, convex optimizations, equilibrium problems, knowledge engineering, etc. For details, we can refer to [1] – [6], [8] – [15] and the references therein. Concerning the development of iterative algorithms for the problem (1.1) in the literature, a very common

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assumption is that T is a maximal monotone operator or m-accretive operator. When T is maximal monotone or m-accretive, many iterative algorithms have been constructed to approximate the solutions of the problem (1.1).

In many practical cases, T is split in the form $T = F + M$, where $F, M : X \to 2^X$ are two multi-valued operators. So the problem (1.1) reduces to the following: Find $x \in X$ such that

$$0 \in F(x) + M(x), \tag{1.2}$$

which is called the variational inclusion problem. When both F and M are maximal monotone or M is m-accretive, some approximate solutions for the problem (1.2) have been developed (see [10, 13] and the references therein). If $M = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a proper convex lower semi-continuous functional $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, then the problem (1.2) reduces to the variational inequality problem:

Find $x \in X$ and $u \in F(x)$ such that

$$\langle u, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad y \in X. \tag{1.3}$$

Many iterative algorithms have been established to approximate the solution of the problem (1.3) when F is strongly monotone. Recently, the problem (1.2) was studied by several authors when F and M need not to be maximal monotone or m-accretive. Further, Ding [3], Huang [6], and Lan et al. [11] developed some iterative algorithms to solve the following quasi-variational inequality problem of finding $x \in X$ and $u \in F(x)$, $v \in V(x)$ such that

$$\langle u, y - x)\rangle + \varphi(y, v) - \varphi(x, v) \geq 0, \quad \forall y \in X \tag{1.4}$$

by introducing the concept of subdifferential $\partial\varphi(\cdot, t)$ of a proper functional $\varphi(\cdot, \cdot)$ for $t \in X$, which is defined by

$$\partial\varphi(\cdot, t) = \{f \in X : \varphi(y, t) - \varphi(x, t) \geq \langle f, y - x)\rangle, \quad y \in X\},$$

where $\varphi(\cdot, t) : X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous functional for all $t \in X$.

It is easy to see that the problem (1.4) is equivalent to the following:

Find $x \in X$ such that

$$0 \in F(x) + \partial\varphi(x, V(x)). \tag{1.5}$$

Recently, Huang and Fang [7] first introduced the concept of a generalized m-accretive mapping, which is a generalization of an m-accretive mapping, and gave the definition and properties of the resolvent operator for the generalized m-accretive mapping in a Banach space. Later, by using the resolvent operator technique, which is a very important method for finding solutions of variational inequality and variational inclusion problems, a number of nonlinear variational inclusions and many systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems. Bi, Huang, Jin and other authors introduced and studied some new classes of nonlinear variational inclusions involving generalized m-accretive mappings in Banach spaces, they also obtained some new corresponding existence and convergence results (see, [2, 5, 8] and the references therein). On the other hand, Huang, Lan, Zeng, Wang et al. discussed the stability of the iterative sequence generated by the algorithm for solving what they studied (see [6, 11, 15, 19]).

Motivated and inspired by the above works, in this paper, we introduce and study the following new class of generalized strongly nonlinear operator inclusion problems involving generalized m-accretive mappings:

Find $x \in X$ such that $(p(x), g(x)) \in \mathrm{Dom}\, M$ and

$$f \in N(S(x), T(x), U(x)) + M(p(x), g(x)), \tag{1.6}$$

where f is an any given element on X, a real Banach space, $S, T, U, p, g : X \to X$ and $N : X \times X \times X \to X$ are single-valued mappings and $M : X \times X \to 2^X$ is a generalized m-accretive mapping with respect to the first argument, 2^X denotes the family of all the nonempty subsets of X. By using the resolvent operator technique for generalized m-accretive mappings due to Huang and Fang [7, 8], we prove the existence theorems of the solution for these types of operator inclusion problems in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm for solving this class of nonlinear operator inclusion problems in Banach spaces. Our results improve and generalize the corresponding results of [3, 6, 11, 12].

We remark that for a suitable choice of f, the mappings $N, \eta, S, T, U, M, p, g$ and the space X, a number of known or new classes of variational inequalities, variational inclusions and corresponding optimization problems can be obtained as special cases of the nonlinear quasi-variational inclusion problem (1.6). Moreover, these classes of variational inclusions provide us with a general and unified framework for studying a wide range of interesting and important problems arising in mechanics, optimization and control, equilibrium theory of transportation and economics, management sciences, and other branches of mathematical and engineering sciences, etc. See for more details [1, 3, 4, 6, 9, 11, 15, 17, 18] and the references therein.

2. GENERALIZED m-ACCRETIVE MAPPING

Throughout this paper, let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex (see, for example, [16]). If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. In what follows we shall denote the single-valued generalized duality mapping by j_q.

Definition 2.1. The mapping $g : X \to X$ is said to be

(1) α-strongly accretive, if for any $x, y \in X$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle g(x) - g(y), j_q(x-y) \rangle \geq \alpha \|x - y\|^q,$$

where $\alpha > 0$ is a constant;

(2) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leq \beta \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2. Let $h, g : X \to X$ be two single-valued mappings. The mapping $N : X \times X \times X \to X$ is said to be

(1) σ-strongly accretive with respect to h in the first argument, if for any $x, y \in X$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle N(h(x), \cdot, \cdot) - N(h(y), \cdot, \cdot), j_q(x-y) \rangle \geq \sigma \|x - y\|^q,$$

where $\sigma > 0$ is a constant;

(2) ς-relaxed accretive with respect to g in the second argument, if for any $x, y \in X$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle N(\cdot, g(x), \cdot) - N(\cdot, g(y), \cdot), j_q(x-y) \rangle \geq -\varsigma \|x - y\|^q$$

where $\varsigma > 0$ is a constant;

(3) ϵ-Lipschitz continuous with respect to the first argument, if there exists a constant $\epsilon > 0$ such that

$$\|N(x,\cdot,\cdot) - N(y,\cdot,\cdot)\| \le \epsilon\|x-y\|, \quad \forall x,y \in X.$$

Similarly, we can define the ξ, γ-Lipschitz continuity in the second and third argument of $N(\cdot,\cdot,\cdot)$, respectively.

Definition 2.3 ([7])**.** Let $\eta:\ X \times X \to X^*$ be a single-valued mapping and $A:\ X \to 2^X$ be a multi-valued mapping. Then A is said to be

(1) η-accretive if

$$\langle u-v, \eta(x,y)\rangle \ge 0, \quad \forall x,y \in X, u \in A(x), v \in A(y);$$

(2) generalized m-accretive if A is η-accretive and $(I+\lambda A)(X) = X$ for all (equivalently, for some) $\lambda > 0$.

Remark 2.1. Huang and Fang gave one example of the generalized m-accretive mapping in [7]. If $X = X^* = H$ is a Hilbert space, then (1), (2) of Definition 2.3 reduce to the definition of η-monotonicity and maximal η-monotonicity respectively; if X is uniformly smooth and $\eta(x,y) = J_2(x-y)$, then (1) and (2) of Definition 2.3 reduce to the definitions of accretivity and m-accretivity in uniformly smooth Banach spaces, respectively (see [7, 8]).

Definition 2.4. The mapping $\eta:\ X \times X \to X^*$ is said to be

(1) δ-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle x-y, \eta(x,y)\rangle \ge \delta\|x-y\|^2, \quad \forall x,y \in X;$$

(2) τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \le \tau\|x-y\|, \quad \forall x,y \in X.$$

The modules of smoothness of X is the function $\rho_X:\ [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\|x+y\| + \|x-y\| - 1:\ \|x\| \le 1,\ \|y\| \le t\right\}.$$

A Banach space X is called uniformly smooth if $\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0$ and X is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X \le ct^q$, where $q > 1$ is a real number.

It is well known that Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [16] proved the following result:

Lemma 2.2. *Let $q > 1$ be a given real number and X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x,y \in X$, $j_q(x) \in J_q(x)$, there holds the following inequality*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

In [7], Huang and Fang show that for any $\rho > 0$, inverse mapping $(I+\rho A)^{-1}$ is single-valued, if $\eta:\ X \times X \to X^*$ is strict monotone and $A:\ X \to 2^X$ is a generalized m-accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [7] gave the following definition:

Definition 2.5. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is defined as follows:

$$J_A^\rho(z) = (I + \rho A)^{-1}(z), \quad \forall z \in X,$$

where $\rho > 0$ is a constant and $\eta : X \times X \to X^*$ is a strictly monotone mapping.

Lemma 2.3 ([7, 8]). *Let $\eta : X \times X \to X^*$ be τ-Lipschitz continuous and δ-strongly monotone. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is Lipschitz continuous with constant $\frac{\tau}{\delta}$, i.e.,*

$$\|J_A^\rho(x) - J_A^\rho(y)\| \le \frac{\tau}{\delta}\|x - y\|, \quad \forall x, y \in X.$$

3. EXISTENCE THEOREMS

In this section, we shall give the existence theorem of problem (1.6). Firstly, from the definition of the resolvent operator for a generalized m-accretive mapping, we have the following lemma:

Lemma 3.1. *x is the solution of problem (1.6) if and only if*

$$p(x) = J^\rho_{M(\cdot,g(x))}[p(x) - \rho(N(S(x), T(x), U(x)) - f)], \tag{3.1}$$

where $J^\rho_{M(\cdot,g(x))} = (I + \rho M(\cdot, g(x)))^{-1}$ and $\rho > 0$ is a constant.

Theorem 3.2. *Let X be a q-uniformly smooth Banach space, $\eta : X \times X \to X^*$ be τ-Lipschitz continuous and δ-strongly monotone, $M : X \times X \to 2^X$ be a generalized m-accretive mapping with respect to the first argument, and mappings $S, T, U : X \to X$ be κ, μ, ν-Lipschitz continuous, respectively. Let $p : X \to X$ be α-strongly accretive and β-Lipschitz continuous, $g : X \to X$ be ι-Lipschitz continuous, $N : X \times X \times X \to X$ be σ-strongly accretive with respect to S in the first argument and ς-relaxed accretive with respect to T in the second argument, and ϵ, ξ, γ-Lipschitz continuous in the first, second and third argument, respectively. Suppose that there exist constants $\rho > 0$ and $\zeta > 0$ such that for each $x, y, z \in X$,*

$$\left\| J^\rho_{M(\cdot,x)}(z) - J^\rho_{M(\cdot,x)}(z) \right\| \le \zeta\|x - y\| \tag{3.2}$$

and

$$\begin{cases} h = \zeta\iota + \left(1 + \frac{\tau}{\delta}\right)(1 - q\alpha + c_q\beta^q)^{\frac{1}{q}} < 1, \\ \tau\left[(1 - q\rho(\sigma - \varsigma) + c_q\rho^q(\epsilon\kappa + \xi\mu)^q)^{\frac{1}{q}} + \rho\gamma\nu\right] < \delta(1 - h), \end{cases} \tag{3.3}$$

where c_q is the same as in Lemma 2.2, then problem (1.6) has a unique solution x^*.

Proof. From Lemma 3.1, problem (1.6) is equivalent to the fixed problem (3.1), equation (3.1) can be rewritten as follows:

$$x = x - p(x) - J^\rho_{M(\cdot,g(x))}[p(x) - \rho(N(S(x), T(x), U(x)) - f)].$$

For every $x \in X$, take

$$Q(x) = x - p(x) - J^\rho_{M(\cdot,g(x))}[p(x) - \rho(N(S(x), T(x), U(x)) - f)]. \tag{3.4}$$

Then x^* is the unique solution of problem (1.6) if and only if x^* is the unique fixed point of Q. In fact, it follows from (3.2), (3.4) and Lemma 2.3 that

$$\begin{aligned}
&\|Q(x)-Q(y)\| \\
&\le \|x-y-(p(x)-p(y))\| + \Big\| J^{\rho}_{M(\cdot,g(x))}[p(x)-\rho(N(S(x),T(x),U(x))-f)] \\
&\qquad - J^{\rho}_{M(\cdot,g(y))}[p(y)-\rho(N(S(y),T(y),U(y))-f)]\Big\| \\
&\le \|x-y-(p(x)-p(y))\| + \Big\| J^{\rho}_{M(\cdot,g(x))}[p(x)-\rho(N(S(x),T(x),U(x))-f)] \\
&\qquad - J^{\rho}_{M(\cdot,g(x))}[p(y)-\rho(N(S(y),T(y),U(y))-f)]\Big\| \\
&\qquad + \Big\| J^{\rho}_{M(\cdot,g(x))}[p(y)-\rho(N(S(y),T(y),U(y))-f)] \\
&\qquad - J^{\rho}_{M(\cdot,g(y))}[p(y)-\rho(N(S(y),T(y),U(y))-f)]\Big\| \\
&\le \left(1+\frac{\tau}{\delta}\right)\|x-y-(p(x)-p(y))\| \\
&\qquad + \frac{\tau}{\delta}\{\|x-y-\rho[(N(S(x),T(x),U(x))-N(S(y),T(x),U(x))) \\
&\qquad + (N(S(y),T(x),U(x))-N(S(y),T(y),U(x)))]\| \\
&\qquad + \rho\|N(S(y),T(y),U(x))-N(S(y),T(y),U(y))\|\} \\
&\qquad + \zeta\|g(x)-g(y))\|.
\end{aligned} \tag{3.5}$$

By the hypothesis of g, p, S,T,U, N and Lemma 2.2, now we know there exists $c_q>0$ such that

$$\|g(x)-g(y)\| \le \iota\|x-y\|, \tag{3.6}$$

$$\|x-y-(p(x)-p(y))\|^q \le (1-q\alpha+c_q\beta^q)\|x-y\|^q, \tag{3.7}$$

$$\|N(S(y),T(y),U(x))-N(S(y),T(y),U(y))\| \le \gamma\nu\|x-y\|, \tag{3.8}$$

$$\begin{aligned}
&\|x-y-\rho[(N(S(x),T(x),U(x))-N(S(y),T(x),U(x))) \\
&\qquad + (N(S(y),T(x),U(x))-N(S(y),T(y),U(x)))]\|^q \\
&\le \|x-y\|^q - q\rho\langle(N(S(x),T(x),U(x))-N(S(y),T(x),U(x))) \\
&\qquad + (N(S(y),T(x),U(x))-N(S(y),T(y),U(x))), j_q(x-y)\rangle \\
&\qquad + c_q\rho^q\|(N(S(x),T(x),U(x))-N(S(y),T(x),U(x))) \\
&\qquad + (N(S(y),T(x),U(x))-N(S(y),T(y),U(x)))\|^q \\
&\le \|x-y\|^q - q\rho[\langle N(S(x),T(x),U(x))-N(S(y),T(x),U(x)), j_q(x-y)\rangle \\
&\qquad + \langle N(S(y),T(x),U(x))-N(S(y),T(y),U(x)), j_q(x-y)\rangle] \\
&\qquad + c_q\rho^q[\|N(S(x),T(x),U(x))-N(S(y),T(x),U(x))\| \\
&\qquad + \|N(S(y),T(x),U(x))-N(S(y),T(y),U(x))\|]^q \\
&\le [1-q\rho(\sigma-\varsigma)+c_q\rho^q(\epsilon\kappa+\xi\mu)^q]\|x-y\|^q.
\end{aligned} \tag{3.9}$$

Combining (3.5) – (3.9), we get

$$\|Q(x)-Q(y)\| \le \theta\|x-y\|, \tag{3.10}$$

where

$$\theta = h + \frac{\tau}{\delta}\left[(1 - q\rho(\sigma - \varsigma) + c_q\rho^q(\epsilon\kappa + \xi\mu)^q)^{\frac{1}{q}} + \rho\gamma\nu\right], \tag{3.11}$$
$$h = \zeta\iota + \left(1 + \frac{\tau}{\delta}\right)(1 - q\alpha + c_q\beta^q)^{\frac{1}{q}}.$$

It follows from (3.3) that $0 < \theta < 1$ and so $Q : X \to X$ is a contractive mapping, i.e., Q has a unique fixed point in X. This completes the proof. □

Remark 3.3. If X is a 2-uniformly smooth Banach space and there exists $\rho > 0$ such that

$$\begin{cases} h = \zeta\iota + \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\alpha + c_2\beta^2} < 1, \\ 0 < \rho < \frac{\delta(1-h)}{\tau\gamma\nu}, \quad \gamma\nu < \sqrt{c_2}(\epsilon\kappa + \xi\mu), \\ \tau(\sigma - \varsigma) > \delta\gamma\nu(1 - h) + \sqrt{[c_2(\epsilon\kappa + \xi\mu)^2 - \gamma^2\nu^2][\tau^2 - \delta^2(1 - h)^2]}, \\ \left|\rho - \frac{\tau(\sigma-\varsigma)+\delta\gamma\nu(h-1)}{\tau[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2]}\right| < \frac{[\tau(\sigma-\varsigma)-\delta\gamma\nu(1-h)]^2-[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2][\tau^2-\delta^2(1-h)^2]}{\tau[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2]}, \end{cases}$$

then (3.3) holds. We note that the Hilbert space and L_p (or l_p) $(2 \leq p < \infty)$ spaces are 2-uniformly Banach spaces.

4. PERTURBED ALGORITHM AND STABILITY

In this section, by using the following definition and lemma, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1.6) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Definition 4.1. Let S be a selfmap of X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 4.1 ([12])**.** *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

there exists a natural number n_0 such that

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0 (n \to \infty)$.

The relation (3.1) allows us to construct the following perturbed iterative algorithm with mixed errors.

Algorithm 4.1. *Step* 1. Choose $x_0 \in X$.

Step 2. Let

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n - p(y_n) \\ \qquad\qquad + J^{\rho}_{M(\cdot,g(y_n))}(p(y_n) - \rho(N(S(y_n), T(y_n), U(y_n)) - f))] + \alpha_n u_n + \omega_n, \\ y_n = (1 - \beta_n)x_n + \beta_n[x_n - p(x_n) \\ \qquad\qquad + J^{\rho}_{M(\cdot,g(x_n))}(p(x_n) - \rho(N(S(x_n), T(x_n), U(x_n)) - f))] + v_n, \end{cases} \tag{4.1}$$

Step 3. Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ such that for $n \geq 0$, $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$, $\{u_n\}, \{v_n\}, \{\omega_n\}$ are sequences in X satisfying the following conditions:

(i) $u_n = u'_n + u''_n$;
(ii) $\lim_{n\to\infty} \|u'_n\| = \lim_{n\to\infty} \|v_n\| = 0$;
(iii) $\sum_{n=0}^{\infty} \|u''_n\| < \infty, \quad \sum_{n=0}^{\infty} \|\omega_n\| < \infty$,

Step 4. If x_{n+1}, y_n, α_n, β_n, u_n, v_n and ω_n satisfy (4.1) to sufficient accuracy, go to *Step* 5; otherwise, set $n := n+1$ and return to *Step* 2.

Step 5. Let $\{z_n\}$ be any sequence in X and define $\{\varepsilon_n\}$ by

$$(4.2)\quad \begin{cases} \varepsilon_n = \|z_{n+1} - \{(1-\alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ \qquad + J^{\rho}_{M(\cdot,g(t_n))}(p(t_n) - \rho(N(S(t_n),T(t_n),U(t_n)) - f))] + \alpha_n u_n + \omega_n\}\|, \\ t_n = (1-\beta_n)z_n + \beta_n[z_n - p(z_n) \\ \qquad + J^{\rho}_{M(\cdot,g(z_n))}(p(z_n) - \rho(N(S(z_n),T(z_n),U(z_n)) - f))] + v_n. \end{cases}$$

Step 6. If ε_n, z_{n+1}, t_n, α_n, β_n, u_n, v_n and ω_n satisfy (4.2) to sufficient accuracy, stop; otherwise, set $n := n+1$ and return to *Step* 3.

Theorem 4.2. *Suppose that $X, S, T, U, p, g, N, \eta$ and M are the same as in Theorem 3.2, θ is defined by (3.11). If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and conditions (3.2), (3.3) hold, then the perturbed iterative sequence $\{x_n\}$ defined by (4.1) converges strongly to the unique solution of problem (1.6). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then $\lim_{n\to\infty} z_n = x^*$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$, where ε_n is defined by (4.2).*

Proof. From Theorem 3.2, we know that problem (1.6) has a unique solution $x^* \in X$. It follows from (4.1), (3.11) and the proof of (3.10) in Theorem 3.2 that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n(\|u'_n\| + \|u''_n\|) + \|\omega_n\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n\|u'_n\| + (\|u''_n\| + \|\omega_n\|). \end{aligned} \tag{4.3}$$

Similarly, we have

$$\|y_n - x^*\| \leq (1 - \beta_n + \beta_n\theta)\|x_n - x^*\| + \|v_n\|. \tag{4.4}$$

Combining (4.3) – (4.4), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| \leq{}& [1 - \alpha_n(1 - \theta(1 - \beta_n + \beta_n\theta))]\|x_n - x^*\| \\ &+ \alpha_n(\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|). \end{aligned} \tag{4.5}$$

Since $\theta < 1$, $0 < \beta_n \leq 1$ $(n \geq 0)$, we have $1-\beta_n+\beta_n\theta < 1$ and $1-\theta(1-\beta_n+\beta_n\theta) > 1-\theta > 0$. Therefore, (4.5) implies

$$\begin{aligned} \|x_{n+1} - x^*\| \leq{}& [1 - \alpha_n(1-\theta)]\|x_n - x^*\| \\ &+ \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|). \end{aligned} \tag{4.6}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows from Lemma 4.1 and (4.6) that $\|x_n - x^*\| \to 0 (n \to \infty)$, i.e., $\{x_n\}$ converges strongly to the unique solution x^* of the problem (1.6).

Now we prove the second conclusion. By (4.2), we know

$$\begin{aligned} \|z_{n+1} - x^*\| \leq{}& \|(1-\alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ &+ J^{\rho}_{M(\cdot,g(t_n))}(p(t_n) - \rho(N(S(t_n),T(t_n),U(t_n)) - f))) \\ &+ \alpha_n u_n + \omega_n - x^*\| + \varepsilon_n. \end{aligned} \tag{4.7}$$

As the proof of inequality (4.6), we have

$$\begin{aligned}(4.8)\quad &\|(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ &\quad+J^{\rho}_{M(\cdot,g(t_n))}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f)))+\alpha_n u_n+\omega_n-x^*\|\\ &\qquad\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &\qquad\quad+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|).\end{aligned}$$

Since $0<a\le\alpha_n$, it follows from (4.7) and (4.8) that

$$\begin{aligned}&\|z_{n+1}-x^*\|\\ &\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|)+\varepsilon_n\\ &\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}\left(\|u_n'\|+\theta\|v_n\|+\frac{\varepsilon_n}{a}\right)+(\|u_n''\|+\|\omega_n\|).\end{aligned}$$

Suppose that $\lim\varepsilon_n=0$. Then from $\sum_{n=0}^{\infty}\alpha_n=\infty$ and Lemma 4.1, we have $\lim z_n=x^*$.

Conversely, if $\lim z_n=x^*$, then we get

$$\begin{aligned}\varepsilon_n&=\|z_{n+1}-\{(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ &\quad+J^{\rho}_{M(\cdot,g(t_n))}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f))]+\alpha_n u_n+\omega_n\}\Big\|\\ &\le\|z_{n+1}-x^*\|+\|(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ &\quad+J^{\rho}_{M(\cdot,g(t_n))}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f)))+\alpha_n u_n+\omega_n-x^*\Big\|\\ &\le\|z_{n+1}-x^*\|+[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &\quad+\alpha_n(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|)\to 0\ (n\to\infty).\end{aligned}$$

This completes the proof. □

Remark 4.3. If $u_n=v_n=\omega_n=0$ $(n\ge 0)$ in Algorithm 4.1, then the conclusions of Theorem 4.2 also hold. The results of Theorems 3.2 and 4.2 improve and generalize the corresponding results of [3, 6, 11, 12].

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