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NEW RESULT IN THE ULTIMATE BOUNDEDNESS OF SOLUTIONS OF A THIRD-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION

M.O. OMEIKE

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AGRICULTURE
ABEOKUTA, NIGERIA.
moomeike@yahoo.com

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ABSTRACT. Sufficient conditions are established for the ultimate boundedness of solutions of certain third-order nonlinear differential equations. Our result improves on Tunc's [C. Tunc, Boundedness of solutions of a third-order nonlinear differential equation, J. Inequal. Pure and Appl. Math., 6(1) Art. 3,2005,1-6].

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1. INTRODUCTION

We consider the third-order nonlinear ordinary differential equation,

$$\dddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x}) + h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

or its equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(x, y, z)z - g(x, y) - h(x, y, z) + p(t, x, y, z), \tag{1.2}$$

where f, g, h and p are continuous in their respective arguments, and the dots denote differentiation with respect to t. The derivatives

$$\frac{\partial f(x,y,z)}{\partial x} \equiv f_x(x,y,z), \quad \frac{\partial f(x,y,z)}{\partial z} \equiv f_z(x,y,z), \quad \frac{\partial h(x,y,z)}{\partial x} \equiv h_x(x,y,z),$$
$$\frac{\partial h(x,y,z)}{\partial y} \equiv h_y(x,y,z), \quad \frac{\partial h(x,y,z)}{\partial z} \equiv h_z(x,y,z) \quad \text{and} \quad \frac{\partial g(x,y)}{\partial x} \equiv g_x(x,y)$$

exist and are continuous. Moreover, the existence and the uniqueness of solutions of (1.1) will be assumed. It is well known that the ultimate boundedness is a very important problem in the theory and applications of differential equations, and an effective method for studying the ultimate boundedness of nonlinear differential equations is still the Lyapunov's direct method (see [1] – [8]).

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Recently, Tunc [7] discussed the ultimate boundedness results of Eq. (1.1) and the following result was proved.

Theorem A (Tunc [7])**.** *Further to the assumptions on the functions f, g, h and p assume the following conditions are satisfied $(a, b, c, l, m$ and $A-$ some positive constants):*

(i) $f(x, y, z) \geq a$ *and* $ab - c > 0$ *for all* x, y, z;
(ii) $\frac{g(x,y)}{y} \geq b$ *for all* $x, y \neq 0$;
(iii) $\frac{h(x,0,0)}{x} \geq c$ *for all* $x \neq 0$;
(iv) $0 < h_x(x, y, 0) < c$, *for all* x, y;
(v) $h_y(x, y, 0) \geq 0$ *for all* x, y;
(vi) $h_z(x, y, 0) \geq m$ *for all* x, y;
(vii) $yf_x(x, y, z) \leq 0, yf_z(x, y, z) \geq 0$ *and* $g_x(x, y) \leq 0$ *for all* x, y, z;
(viii) $yzh_y(x, y, 0) + ayzh_z(x, y, z) \geq 0$ *for all* x, y, z;
(ix) $|p(t, x, y, z)| \leq e(t)$ *for all* $t \geq 0, x, y, z$,
where $\int_0^t e(s)ds \leq A < \infty$.

Then, given any finite numbers x_0, y_0, z_0 there is a finite constant $D = D(x_0, y_0, z_0)$ such that the unique solution $(x(t), y(t), z(t))$ of (1.2) which is determined by the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0$$

satisfies

$$|x(t)| \leq D, \qquad |y(t)| \leq D, \qquad |z(t)| \leq D$$

for all $t \geq 0$.

Theoretically, this is a very interesting result since (1.1) is a rather general third-order nonlinear differential equation. For example, many third order differential equations which have been discussed in [5] are special cases of Eq. (1.1), and some known results can be obtained by using this theorem. However, it is not easy to apply Theorem A to these special cases to obtain new or better results since Theorem A has some hypotheses which are not necessary for the stability of many nonlinear equations. The Lyapunov function used in the proof of Theorem A is not complete (see [2]). Furthermore, the boundedness result considered in [7] is of the type in which the bounding constant depends on the solution in question.

Our aim in this paper is to further study the boundedness of solutions of Eq. (1.1). In the next section, we establish a criterion for the ultimate boundedness of solutions of Eq. (1.1), which extends and improves Theorem A.

Our main result is the following theorem.

Theorem 1.1. *Further to the basic assumptions on the functions f, g, h and p assume that the following conditions are satisfied (a, b, c, ν and $A-$ some positive constants):*

(i) $f(x, y, z) > a$ *and* $ab - c > 0$ *for all* x, y, z;
(ii) $\frac{g(x,y)}{y} \geq b$ *for all* $x, y \neq 0$;
(iii) $\frac{h(x,y,z)}{x} \geq \nu$ *for all* $x \neq 0$;
(iv) $h_x(x, 0, 0) \leq c$, $h_y(x, y, 0) \geq 0$ *and* $h_z(x, 0, z) \geq 0$ *for all* x, y, z;
(v) $yf_x(x, y, z) \leq 0, yf_z(x, y, z) \geq 0$ *and* $g_x(x, y) \leq 0$ *for all* x, y, z;
(vi) $|p(t, x, y, z)| \leq A < \infty$ *for all* $t \geq 0$.

Then every solution $x(t)$ of (1.1) satisfies

$$|x(t)| \leq D, \qquad |\dot{x}(t)| \leq D, \qquad |\ddot{x}(t)| \leq D \tag{1.3}$$

for all sufficiently large t, where D is a constant depending only on a, b, c, A and ν.

2. PRELIMINARIES

It is convenient here to consider, in place of the equation (1.1), the system (1.2). It is to be shown then, in order to prove the theorem, that, under the conditions stated in the theorem, every solution $(x(t), y(t), z(t))$ of (1.2) satisfies

$$|x(t)| \le D, \qquad |y(t)| \le D, \qquad |z(t)| \le D \tag{2.1}$$

for all sufficiently large t, where D is the constant in (1.3).

Our proof of (2.1) rests entirely on two properties (stated in the lemma below) of the function $V = V(x, y, z)$ defined by

$$V = V_1 + V_2, \tag{2.2}$$

where V_1, V_2 are given by

$$\begin{aligned}2V_1 = 2\int_0^x h(\xi,0,0)d\xi + 2\int_0^y \eta f(x,\eta,0)d\eta + 2\delta\int_0^y g(x,\eta)d\eta \\ + \delta z^2 + 2yz + 2\delta y h(x,0,0) - \alpha\beta y^2,\end{aligned} \tag{2.3a}$$

$$\begin{aligned}2V_2 = \alpha\beta b x^2 + 2a\int_0^x h(\xi,0,0)d\xi + 2a\int_0^y \eta f(x,\eta,0)d\eta \\ + 2\int_0^y g(x,\eta)d\eta + z^2 + 2a\alpha\beta xy + 2\alpha\beta xz + 2ayz + 2yh(x,0,0),\end{aligned} \tag{2.3b}$$

where $\frac{1}{a} < \delta < \frac{b}{c}$, and α, β are some positive constants such that

$$\alpha < \min\left\{ \frac{ab-c}{\beta\left[a + \nu^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]}; \frac{1}{a}; \frac{a\delta - 1}{ab\delta}; \frac{\nu(a\delta-1)}{\beta[f(x,y,z)-a]^2} \right\}$$

and β will be fixed to advantage later.

Lemma 2.1. *Subject to the conditions of Theorem 1.1, $V(0,0,0) = 0$ and there is a positive constant D_1 depending only on a, b, c, α and δ such that*

$$V(x,y,z) \ge D_1(x^2 + y^2 + z^2) \tag{2.4}$$

for all x, y, z. Furthermore, there are finite constants $D_2 > 0$, $D_3 > 0$ dependent only on a, b, c, A, ν, δ, and α such that for any solution $(x(t), y(t), z(t))$ of (1.2),

$$\dot{V} \equiv \frac{d}{dt}V(x(t), y(t), z(t)) \le -D_2, \tag{2.5}$$

provided that $x^2 + y^2 + z^2 \ge D_3$.

Proof of Lemma 2.1. To verify (2.4) observe first that the expressions (2.3) defining $2V_1, 2V_2$ may be rewritten in the forms

$$\begin{aligned}2V_1 = \left\{2\int_0^x h(\xi,0,0)d\xi - \frac{\delta}{b}h^2(x,0,0)\right\} + \delta b\left\{y + \frac{h(x,0,0)}{b}\right\}^2 \\ + \left\{2\int_0^y \eta f(x,\eta,0)d\eta - \delta^{-1}y^2 - \alpha\beta y^2\right\} + \delta(z + \delta^{-1}y)^2 \\ + \delta\left\{2\int_0^y g(x,\eta)d\eta - by^2\right\}\end{aligned}$$

and

$$2V_2 = \alpha\beta(b-\alpha\beta)x^2 + a\left\{2\int_0^x h(\xi,0,0)d\xi - \beta^{-1}h^2(x,0,0)\right\}$$
$$+ \beta\left\{a^{-\frac{1}{2}}y + \beta^{-1}a^{\frac{1}{2}}h(x,0,0)\right\}^2 + \left\{2\int_0^y g(x,\eta)d\eta - \beta a^{-1}y^2\right\}$$
$$+ a\left\{2\int_0^y \eta f(x,\eta,0)d\eta - ay^2\right\} + (\alpha\beta x + ay + z)^2.$$

The term $2\int_0^x h(\xi,0,0)d\xi - \frac{\delta}{b}h^2(x,0,0)$ in the rearrangement for $2V_1$ is evidently equal to

$$2\int_0^x \left\{1 - \frac{\delta}{b}h_\xi(\xi,0,0)\right\} h(\xi,0,0)d\xi - \frac{\delta}{b}h^2(0,0,0).$$

By conditions (iii) and (iv) of Theorem 1.1 and $h(0,0,0)=0$, we have

$$2\int_0^x \left\{1 - \frac{\delta}{b}h_\xi(\xi,0,0)\right\} h(\xi,0,0)d\xi - \frac{\delta}{b}h^2(0,0,0) \geq \left(1-\frac{\delta}{b}c\right)\nu x^2.$$

In the same way, using (iii) and (iv), it can be shown that the term

$$\left\{2\int_0^x h(\xi,0,0)d\xi - \beta^{-1}h^2(x,0,0)\right\}$$

appearing in the rearrangement for $2V_2$ satisfies

$$\left\{2\int_0^x h(\xi,0,0)d\xi - \beta^{-1}h^2(x,0,0)\right\} \geq \left(1-\frac{c}{\beta}\right)\nu x^2,$$

for all x.

Since $\frac{h(x,y,z)}{x} \geq \nu \ \ (x \neq 0)$, $\ \frac{g(x,y)}{y} \geq b$, $\ (y \neq 0)$ and $f(x,y,z) > a$, and combining all these with (2.2), we have

$$2V \geq \left\{\nu\left(1-\frac{\delta}{b}c\right) + \alpha\beta(b-\alpha\beta) + a\nu\left(1-\frac{c}{\beta}\right)\right\}x^2$$
$$+ \left\{\left(a - \frac{1}{\delta} - \alpha\beta\right) + \left(b - \frac{\beta}{a}\right)\right\}y^2 + \delta\left(z + \frac{1}{\delta}y\right)^2 + (\alpha\beta x + ay + z)^2$$

for all x, y and z. Hence if we choose $\beta = ab$ the constants $1-\frac{\delta}{b}c$, $b-\alpha\beta$, $1-\frac{c}{\beta}$, $a-\frac{1}{\delta}-\alpha\beta$ and $b-\frac{\beta}{a}$ are either zero or positive. This implies that there exists a constant D_1 small enough such that (2.4) holds.

To deal with the other half of the lemma, let $(x(t), y(t), z(t))$ be any solution of (1.2) and consider the function

$$V(t) \equiv V\left(x(t), y(t), z(t)\right).$$

By an elementary calculation using (1.2), (2.2) and (2.3), we have that

$$\begin{aligned}(2.6)\quad \dot{V} &= (1+\delta)y\int_0^y g_x(x,\eta)d\eta + (1+a)y\int_0^y \eta f_x(x,\eta,0)d\eta \\ &\quad - (1+a)\frac{\{f(x,y,z)-f(x,y,0)\}}{z}yz^2 - (1+a)\frac{\{h(x,y,z)-h(x,0,0)\}}{y}y^2 \\ &\quad - (1+\delta)\frac{\{h(x,y,z)-h(x,0,0)\}}{z}z^2 - \alpha\beta\frac{h(x,y,z)}{x}x^2 - \frac{g(x,y)}{y}y^2 \\ &\quad - a\frac{g(x,y)}{y}y^2 + \delta h_x(x,0,0)y^2 + h_x(x,0,0)y^2 + a\alpha\beta y^2 \\ &\quad - \delta f(x,y,z)z^2 - [f(x,y,z)-a]z^2 + z^2 - \alpha\beta\left\{\frac{g(x,y)}{y} - b\right\}xy \\ &\quad - \alpha\beta\{f(x,y,z)-a\}xz + \{\alpha\beta x + (1+a)y + (1+\delta)z\}p(t,x,y,z).\end{aligned}$$

By (v), we get

$$y\int_0^y g_x(x,\eta)d\eta \le 0, \qquad y\int_0^y f_x(x,\eta,0)\eta d\eta \le 0.$$

It follows from (v), for $z \neq 0$ that

$$W_1 = a\frac{\{f(x,y,z)-f(x,y,0)\}}{z}yz^2 = af_z(x,y,\theta_1 z)yz^2 \ge 0,$$

$0 \le \theta_1 \le 1$ but $W_1 = 0$ when $z = 0$. Hence

$$W_1 \ge 0 \quad \text{for all } x, y, z.$$

Similarly, it is clear that

$$W_2 = \frac{\{h(x,y,z)-h(x,0,0)\}}{y}y^2 = h_y(x,\theta_2 y,0)y^2 \ge 0,$$

$0 \le \theta_2 \le 1$ but $W_2 = 0$ when $y = 0$. Hence

$$W_2 \ge 0 \quad for \;\; all \;\; x, y.$$

Also,

$$W_3 = \frac{\{h(x,y,z)-h(x,0,0)\}}{z}z^2 = h_z(x,0,\theta_3 z)z^2 \ge 0,$$

$0 \le \theta_3 \le 1$ but $W_3 = 0$ when $z = 0$. Hence

$$W_3 \ge 0 \quad \text{for all } x, z.$$

Then, combining the estimates W_1, W_2, W_3 and (iii) with (2.6) we obtain

$$\begin{aligned}
\dot{V} &\leq -\alpha\beta\nu x^2 - (ab - c - \alpha\beta a)y^2 - (b - \delta c)y^2 - (a\delta - 1)z^2 \\
&\qquad - az^2 - \alpha\beta \left\{ \frac{g(x,y)}{y} - b \right\} xy - \alpha\beta\{f(x,y,z) - a\}xz \\
&\qquad + \{\alpha\beta x + (1+a)y + (1+\delta)z\}p(t,x,y,z) \\
&= -\frac{1}{2}\alpha\beta\nu x^2 - \left\{ ab - c - \alpha\beta \left[a + \nu^{-1} \left(\frac{g(x,y)}{y} - b \right)^2 \right] \right\} y^2 \\
&\qquad - (b - \delta c)y^2 - \left\{ a\delta - 1 - \alpha\beta\nu^{-1}[f(x,y,z) - a]^2 \right\} z^2 - az^2 \\
&\qquad - \frac{1}{4}\alpha\beta\nu \left\{ \left[x + 2\nu^{-1} \left(\frac{g(x,y)}{y} - b \right) y \right]^2 + \left[x + 2\nu^{-1}(f(x,y,z) - a)z \right]^2 \right\} \\
&\qquad + \{\alpha\beta x + (1+a)y + (1+\delta)z\}p(t,x,y,z).
\end{aligned}$$

If we choose

$$\alpha < \min \left\{ \frac{ab - c}{\beta \left[a + \nu^{-1} \left(\frac{g(x,y)}{y} - b \right)^2 \right]}; \frac{1}{a}; \frac{a\delta - 1}{ab\delta}; \frac{\nu(a\delta - 1)}{\beta[f(x,y,z) - a]^2} \right\},$$

it follows that

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{2}\alpha\beta\nu x^2 - (b - \delta c)y^2 - az^2 + \{\alpha\beta x + (1+a)y + (1+\delta)z\}p(t,x,y,z) \\
&\leq -D_4(x^2 + y^2 + z^2) + D_5(|x| + |y| + |z|),
\end{aligned}$$

where

$$D_4 = \min \left\{ \frac{1}{2}\alpha\beta\nu; b - \delta c; a \right\}, \qquad D_5 = A \max\{\alpha\beta; 1 + a; 1 + \delta\}.$$

Moreover,

$$\dot{V} \leq -D_4(x^2 + y^2 + z^2) + D_6(x^2 + y^2 + z^2)^{\frac{1}{2}}, \tag{2.7}$$

where $D_6 = 3^{\frac{1}{2}} D_5$.

If we choose $(x^2 + y^2 + z^2)^{\frac{1}{2}} \geq D_7 = 2D_6 D_4^{-1}$, inequality (2.7) implies that

$$\dot{V} \leq -\frac{1}{2}D_4(x^2 + y^2 + z^2).$$

We see at once that

$$\dot{V} \leq -D_8,$$

provided that $x^2 + y^2 + z^2 \geq 2D_8 D_4^{-1}$; and this completes the verification of (2.5). □

3. PROOF OF THEOREM 1.1

Let $(x(t), y(t), z(t))$ be any solution of (1.2). Then there is evidently a $t_0 \geq 0$ such that

$$x^2(t_0) + y^2(t_0) + z^2(t_0) < D_3,$$

where D_3 is the constant in the lemma; for otherwise, that is if

$$x^2(t) + y^2(t) + z^2(t) \geq D_3, \quad t \geq 0,$$

then, by (2.5),

$$\dot{V}(t) \leq -D_2 < 0, \quad t \geq 0,$$

and this in turn implies that $V(t) \to -\infty$ as $t \to \infty$, which contradicts (2.4). Hence to prove (1.3) it will suffice to show that if

$$x^2(t) + y^2(t) + z^2(t) < D_9 \quad \text{for} \quad t = T, \tag{3.1}$$

where $D_9 \geq D_3$ is a finite constant, then there is a constant $D_{10} > 0$, depending on a, b, c, δ, α and D_9, such that

$$x^2(t) + y^2(t) + z^2(t) \leq D_{10} \quad \text{for} \quad t \geq T. \tag{3.2}$$

Our proof of (3.2) is based essentially on an extension of an argument in the proof of [8, Lemma 1]. For any given constant $d > 0$ let $S(d)$ denote the surface: $x^2 + y^2 + z^2 = d$. Because V is continuous in x, y, z and tends to $+\infty$ as $x^2 + y^2 + z^2 \to \infty$, there is evidently a constant $D_{11} > 0$, depending on D_9 as well as on a, b, c, δ and α, such that

$$\min_{(x,y,z)\in S(D_{11})} V(x, y, z) > \max_{(x,y,z)\in S(D_9)} V(x, y, z). \tag{3.3}$$

It is easy to see from (3.1) and (3.3) that

$$x^2(t) + y^2(t) + z^2(t) < D_{11} \quad \text{for} \quad t \geq T. \tag{3.4}$$

For suppose on the contrary that there is a $t > T$ such that

$$x^2(t) + y^2(t) + z^2(t) \geq D_{11}.$$

Then, by (3.1) and by the continuity of the quantities $x(t), y(t), z(t)$ in the argument displayed, there exist $t_1, t_2, T < t_1 < t_2$ such that

$$x^2(t_1) + y^2(t_1) + z^2(t_1) = D_9, \tag{3.5a}$$

$$x^2(t_2) + y^2(t_2) + z^2(t_2) = D_{11} \tag{3.5b}$$

and such that

$$D_9 \leq x^2(t) + y^2(t) + z^2(t) \leq D_{11}, \quad t_1 \leq t \leq t_2. \tag{3.6}$$

But, writing $V(t) \equiv V\left(x(t), y(t), z(t)\right)$, since $D_9 \geq D_3$, (3.6) obviously implies [in view of (2.5)] that

$$V(t_2) < V(t_1)$$

and this contradicts the conclusion [from (3.3) and (3.5)]:

$$V(t_2) > V(t_1).$$

Hence (3.4) holds. This completes the proof of (1.3), and the theorem now follows.

Remark 3.1. Clearly, our theorem is an improvement and extension of Theorem A. In particular, from our theorem we see that (viii) assumed in Theorem A is not necessary, and (iv) and (ix) can be replaced by $h_x(x, 0, 0) \leq c$ and (vi) of Theorem 1.1 respectively, for the ultimate boundedness of the solutions of Eq. (1.1).

Remark 3.2. Clearly, unlike in [7], the bounding constant D in Theorem 1.1 does not depend on the solution of (1.1).

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