TURÁN-TYPE INEQUALITIES FOR SOME q-SPECIAL FUNCTIONS

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Received: 24 March, 2008
Accepted: 10 April, 2009
Communicated by: S.S. Dragomir
2000 AMS Sub. Class.: 26D07, 26D15, 33D05.
Key words: Turan's inequality, q-polygamma functions, q-zeta function.

Abstract: In this paper, we give new Turán-type inequalities for some q-special functions, using a q- analogue of a generalization of the Schwarz inequality.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

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Kamel Brahim

vol. 10, iss. 2, art. 50, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

In [9], P. Turán proved that the Legendre polynomials $P_n(x)$ satisfy the inequality

$$P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0, \quad x \in [-1, 1], \quad n = 0, 1, 2, \ldots \tag{1.1}$$

and equality occurs only if $x = \pm 1$.

This inequality been the subject of much attention and several authors have provided new proofs, generalizations, extensions and refinements of (1.1).

In [7], A. Laforgia and P. Natalini established some new Turán-type inequalities for polygamma and Riemann zeta functions:

Theorem 1.1. *For $n = 1, 2, \ldots$ we denote by $\psi_n(x) = \psi^{(n)}(x)$ the polygamma functions defined as the $n-th$ derivative of the psi function*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0$$

with the usual notation for the gamma function. Then

$$\psi_m(x)\psi_n(x) \geq \psi_{\frac{m+n}{2}}^2(x),$$

where $\frac{m+n}{2}$ is an integer

Theorem 1.2. *We denote by $\zeta(s)$ the Riemann zeta function. Then*

$$(s+1)\frac{\zeta(s)}{\zeta(s+1)} \geq s\frac{\zeta(s+1)}{\zeta(s+2)}, \quad \forall s > 1. \tag{1.2}$$

The main aim of this paper is to give some new Turán-type inequalities for the q-polygamma and q-zeta [2] functions by using a q-analogue of the generalization of the Schwarz inequality.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

To make the paper more self contained we begin by giving some usual notions and notations used in q-theory. Throughout this paper we will fix $q \in]0,1[$ and adapt the notations of the Gasper-Rahman book [4].

Let a be a complex number, the q-shifted factorial are defined by:

$$(a;q)_0 = 1; \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \qquad n = 1,2,\dots \tag{1.3}$$

$$(a;q)_\infty = \lim_{n\to+\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.4}$$

For x complex we denote

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.5}$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by [4, 5]:

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n \tag{1.6}$$

and

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n \tag{1.7}$$

provided the sums converge absolutely.

Jackson [5] defined the q-analogue of the Gamma function as:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \qquad x \neq 0,-1,-2,\dots. \tag{1.8}$$

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issn: 1443-5756

It satisfies the functional equation:

(1.9) $$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1$$

and tends to $\Gamma(x)$ when q tends to 1.

Moreover, it has the q-integral representation (see [1, 3])

$$\Gamma_q(s) = K_q(s) \int_0^\infty x^{s-1} e_q^{-x} d_q x,$$

where

$$e_q^x = \frac{1}{((1-q)x;q)_\infty},$$

and

$$K_q(t) = \frac{(1-q)^{-s}}{1+(1-q)^{-1}} \cdot \frac{(-(1-q),q)_\infty (-(1-q)^{-1},q)_\infty}{(-(1-q)q^s,q)_\infty (-(1-q)^{-1}q^{1-s},q)_\infty}.$$

Lemma 1.3. *Let $a \in \mathbb{R}_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then*

(1.10) $$\left(\int_0^a g(x) f^{\frac{m+n}{2}}(x) d_q x\right)^2 \leq \left(\int_0^a g(x) f^m(x) d_q x\right)\left(\int_0^a g(x) f^n(x) d_q x\right),$$

where m and n belong to a set S of real numbers, such that the integrals (1.10) exist.

Proof. Letting $a > 0$, by definition of the q-Jackson integral, we have

(1.11) $$\int_0^a g(x) f^{\frac{m+n}{2}}(x) d_q x = (1-q)a \sum_{p=0}^\infty g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p$$

$$= \lim_{N \to +\infty} (1-q)a \sum_{p=0}^N g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756
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vol. 10, iss. 2, art. 50, 2009

By the use of the Schwarz inequality for finite sums, we obtain

$$\left(\sum_{p=0}^{N} g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p\right)^2 \leq \left(\sum_{p=0}^{N} g(aq^p) f^m(aq^p) q^p\right)\left(\sum_{p=0}^{N} g(aq^p) f^n(aq^p) q^p\right). \tag{1.12}$$

The result follows from the relation (1.11) and (1.12).

To obtain the inequality for $a = \infty$, it suffices to write the inequality (1.10) for $a = q^{-N}$, then tend N to ∞. □

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issn: 1443-5756

2. The q-Polygamma Functions

The q-analogue of the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is defined as the logarithmic derivative of the q-gamma function, $\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$.

From (1.8), we get for $x > 0$

$$\begin{aligned}\psi_q(x) &= -\operatorname{Log}(1-q) + \operatorname{Log} q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} \\ &= -\operatorname{Log}(1-q) + \operatorname{Log} q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^{n}}.\end{aligned}$$

The last equality implies that

$$\psi_q(x) = -\operatorname{Log}(1-q) + \frac{\operatorname{Log} q}{1-q} \int_0^q \frac{t^{x-1}}{1-t} d_q t. \tag{2.1}$$

Theorem 2.1. *For $n = 1, 2, \ldots$, put $\psi_{q,n} = \psi_q^{(n)}$ the n-th derivative of the function ψ_q. Then*

$$\psi_{q,n}(x)\psi_{q,m}(x) \geq \psi^2_{q,\frac{m+n}{2}}(x), \tag{2.2}$$

where $\frac{m+n}{2}$ is an integer.

Proof. Let m and n be two integers of the same parity.

From the relation (2.1) we deduce that

$$\psi_{q,n}(x) = \frac{\operatorname{Log} q}{1-q} \int_0^q \frac{(\operatorname{Log} t)^n t^{x-1}}{1-t} d_q t.$$

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issn: 1443-5756

Applying Lemma 1.3 with $g(t) = \frac{t^{x-1}}{1-t}$, $f(t) = (-\operatorname{Log} t)$ and $a = q$, we obtain

$$\int_0^q \frac{t^{x-1}}{1-t}(-\operatorname{Log} t)^n d_q t \int_0^q \frac{t^{x-1}}{1-t}(-\operatorname{Log} t)^m d_q t \geq \left[\int_0^q \frac{t^{x-1}}{1-t}(-\operatorname{Log} t)^{\frac{m+n}{2}} d_q t\right]^2, \tag{2.3}$$

which gives, since $m + n$ is even,

$$\psi_{q,n}(x)\psi_{q,m}(x) \geq \psi^2_{q,\frac{m+n}{2}}(x). \tag{2.4}$$

□

Taking $m = n + 2$, one obtains:

Corollary 2.2. *For all $x > 0$ we have*

$$\frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)} \geq \frac{\psi_{q,n+1}(x)}{\psi_{q,n+2}(x)}, \quad n = 1, 2, \ldots. \tag{2.5}$$

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issn: 1443-5756

3. The q- Zeta function

For $x > 0$, we put

$$\alpha(x) = \frac{\mathrm{Log}(x)}{\mathrm{Log}(q)} - E\left(\frac{\mathrm{Log}(x)}{\mathrm{Log}(q)}\right)$$

and

$$\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}},$$

where $E\left(\frac{\mathrm{Log}(x)}{\mathrm{Log}(q)}\right)$ is the integer part of $\frac{\mathrm{Log}(x)}{\mathrm{Log}(q)}$.

In [2], the authors defined the q-Zeta function as follows

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}. \tag{3.1}$$

They proved that it is a q-analogue of the classical Riemann Zeta function and we have for all $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$\zeta_q(s) = \frac{1}{\widetilde{\Gamma}_q(s)} \int_0^{\infty} t^{s-1} Z_q(t) d_q t,$$

where for all $t > 0$,

$$Z_q(t) = \sum_{n=1}^{\infty} e_q^{-\{n\}_q t} \quad \text{and} \quad \widetilde{\Gamma}_q(t) = \frac{\Gamma_q(t)}{K_q(t)}.$$

Theorem 3.1. *For all $s > 1$ we have*

$$[s+1]_q \frac{\zeta_q(s)}{\zeta_q(s+1)} \geq q[s]_q \frac{\zeta_q(s+1)}{\zeta_q(s+2)}. \tag{3.2}$$

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issn: 1443-5756

Proof. For $s > 1$ the function q-zeta satisfies the following relation

$$\zeta_q(s) = \frac{1}{\widetilde{\Gamma}_q(s)} \int_0^\infty t^{s-1} Z_q(t) d_q t. \tag{3.3}$$

Applying Lemma 1.3 with $g(t) = Z_q(t)$, $f(t) = t$ we obtain

$$\int_0^\infty t^{s-1} Z_q(t) d_q t \int_0^\infty t^{s+1} Z_q(t) d_q t \geq \left[\int_0^\infty t^s Z_q(t) d_q t \right]^2. \tag{3.4}$$

Further, using (3.3), this inequality becomes

$$\zeta_q(s)\widetilde{\Gamma}_q(s)\zeta_q(s+2)\widetilde{\Gamma}_q(s+2) \geq [\zeta_q(s+1)]^2 \left[\widetilde{\Gamma}_q(s+1)\right]^2. \tag{3.5}$$

So, by using the relation $\widetilde{\Gamma}_q(s+1) = q^{-s}[s]_q \widetilde{\Gamma}_q(s)$, we obtain

$$[s+1]_q \zeta_q(s)\zeta_q(s+2) \geq q[s]_q \left[\zeta_q(s+1)\right]^2 \tag{3.6}$$

which completes the proof. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] A.De SOLE AND V.G. KAC, On integral representations of q-gamma and q-beta function, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl.*, **16**(9) (2005), 11–29.

[2] A. FITOUHI, N. BETTAIBI AND K. BRAHIM, The Mellin transform in quantum calculus, *Constructive Approximation*, **23**(3) (2006), 305–323.

[3] A. FITOUHI AND K. BRAHIM, Tauberian theorems in quantum calculus, *J. Nonlinear Mathematical Physics*, **14**(3) (2007), 316–332.

[4] G. GASPER AND M. RAHMAN, Basic hypergeometric series, *Encyclopedia of Mathematics and its Applications*, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1990.

[5] F.H. JACKSON, On a q-definite integral, *Quarterly J. Pure and Appl. Math.*, **41** (1910), 193–203.

[6] V.G. KAC AND P. CHEUNG, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[7] A. LAFORGIA AND P. NATALINI, Turán -type inequalities for some special functions, *J. Ineq. Pure. App. Math.*, **7**(1) (2006), Art. 22. [ONLINE: http://jipam.vu.edu.au/article.php?sid=638]

[8] G. SZEGŐ, On an inequality of P. Turán concerning Legendre polynomials, *Bull. Amer. Math. Soc.*, **54** (1948), 401–405.

[9] P. TURÁN, On the zeros of the polynomials of Legendre, *Casopis pro Pestovani Mat. a Fys*, **75** (1950), 113–122.

vol. 10, iss. 2, art. 50, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756