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CERTAIN INEQUALITIES CONCERNING SOME KINDS OF CHORDAL POLYGONS

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ABSTRACT. This paper deals with certain inequalities concerning some kinds of chordal polygons (Definition 1.2). The main part of the article concerns the inequality

$$\sum_{j=1}^{n} \cos \beta_j > 2k,$$

where

$$\sum_{j=1}^{n} \beta_j = (n-2k)\frac{\pi}{2}, \quad n-2k>0, \qquad 0<\beta_j<\frac{\pi}{2}, \quad j=\overline{1,n}.$$

This inequality is considered and proved in [5, Theorem 1, pp.143-145]. Here we have obtained some new results. Among others we found some chordal polygons with the property that $\sum_{j=1}^{n} \cos^2 \beta_j = 2k$, where $n=4k$ (Theorem 2.17). Also it could be mentioned that Theorem 2.19 is a modest generalization of the Pythagorean theorem.

Key words and phrases: Inequality, k-chordal polygon, k-inscribed chordal polygon, index of k-inscribed chordal polygon, characteristic of k-chordal polygon.

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1. INTRODUCTION

To begin, we will quote some results given in [5], [6].

A polygon with vertices $A_1, \ldots, A_n$ (in this order) will be denoted by $\mathcal{A} \equiv A_1 \cdots A_n$ and the lengths of its sides we will denote by $a_1, \ldots, a_n$. The interior angle at the vertex A_j will be signed by α_j or $\sphericalangle A_j$. Thus

$$\sphericalangle A_j = \sphericalangle A_{j-1}A_jA_{j+1}, \qquad j=\overline{1,n},$$

where $A_0 = A_n$ and $A_{n+1} = A_1$.

A polygon $\mathcal{A}$ is called a chordal polygon if there exists a circle $\mathcal{K}$ such that $A_j \in \mathcal{K},\ j=\overline{1,n}$.

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Remark 1.1. We shall assume that the considered chordal polygon has the property that no two of its consecutive vertices are the same.

For $\mathcal{A}$ chordal, by C and r we denote its centre and the radius of its circumcircle $\mathcal{K}$ respectively.

A very important role will be played by the angles

(1.1) $$\beta_j = \sphericalangle CA_jA_{j+1},$$

(1.2) $$\varphi_j = \sphericalangle A_jCA_{j+1}, \qquad j = \overline{1,n}.$$

We shall use oriented angles, as it is known, an angle $\sphericalangle PQR$ is positively or negatively oriented if it is going from QP to QR counter-clockwise or clockwise. It is very important to emphasize that the angles $\beta_j,\ \varphi_j$ have opposite orientations, see e.g. Fig. 1.1. Of course, the measure of

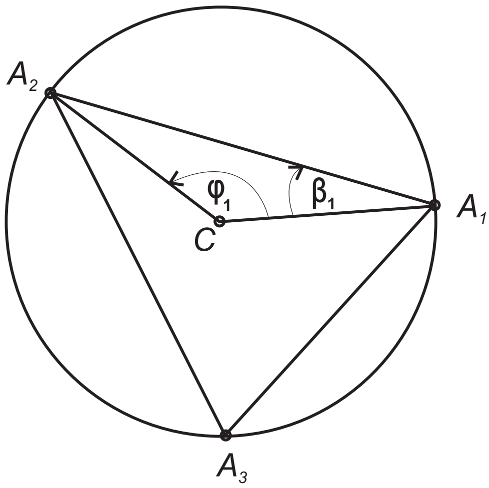

Figure 1.1:

an oriented angle will be taken with $+$ or $-$ depending on whether the angle is positively or negatively oriented. The measure of an angle will usually be expressed by radians.

Remark 1.2. For the sake of simplicity, we shall also write the measures of the oriented angles in (1.1) and (1.2) as $\beta_j,\ \varphi_j$. Obviously, for all $\beta_j,\ \varphi_j$ the following is valid

$$0 \le |\beta_j| < \frac{\pi}{2}, \qquad 0 < |\varphi_j| \le \pi,$$

since no two consecutive vertices in $A_1 \cdots A_n$ are the same, compare Remark 1.1.

Remark 1.3. In the sequel, unless specified otherwise, we shall suppose that no $\beta_j = 0$, i.e.

$$0 < |\beta_j| < \frac{\pi}{2}, \quad j = \overline{1,n}.$$

Accordingly, in the sequel when we refer to chordal polygons, it will be meant (by Remark 1.1 and Remark 1.2) that the polygon has no two consecutive overlapping vertices and no one of its sides is its diameter.

Definition 1.1. Let $\mathcal{A}$ be a chordal polygon. We say that $\mathcal{A}$ is of the *first kind* if inside of $\mathcal{A}$ there is a point O such that all oriented angles $\sphericalangle A_jOA_{j+1},\ j = \overline{1,n}$ have the same orientation. If such a point O does not exist, we say that $\mathcal{A}$ is of the *second kind*.

Definition 1.2. Let $\mathcal{A}$ be a chordal polygon and let $O \in \mathrm{Int}(\mathcal{A})$, such that

$$|\psi_1 + \cdots + \psi_n| = 2k\pi,$$

where $\psi_j =$ measure of the oriented angle $\sphericalangle A_jOA_{j+1}$ and k is a positive integer. Then $\mathcal{A}$ is called a k-inscribed *chordal polygon* or, for brevity, $k-$inscribed polygon if O is such a point that k is maximal, i.e. no other interior point P exists such that $k < m$ and at the same time the following is valid

$$|\psi_1 + \cdots + \psi_n| = 2m\pi,$$

where now $\psi_j =$ measure of the oriented angle $\sphericalangle A_jPA_{j+1}$.

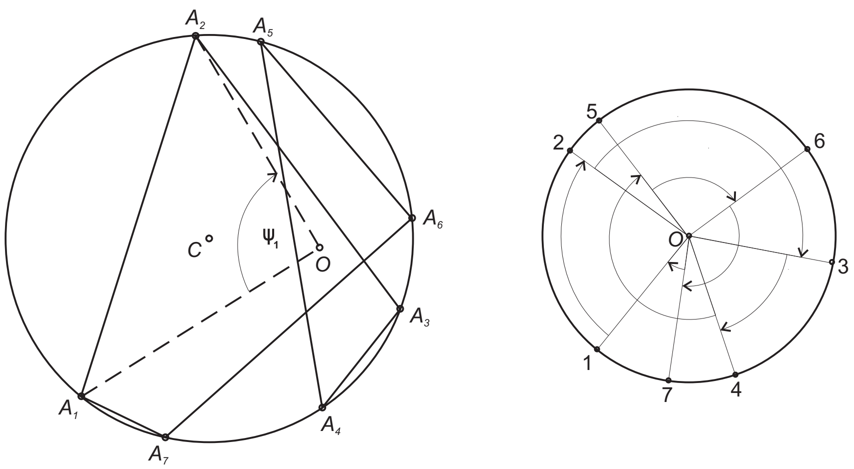

Figure 1.2:

For example, the heptagon $A_1 \cdots A_7$ drawn in Fig. 1.2 is 2-inscribed chordal, since $|\psi_1 + \cdots + \psi_7| = 4\pi$. This heptagon is, according to Definition 1.1, of the first kind – all angles ψ_j have the same, negative orientation.

Of course, a k-inscribed polygon is of the second kind if not all angles ψ_j have the same orientation.

Definition 1.3. Let $\mathcal{A}$ be a k-inscribed chordal n-gon and let

$$|\varphi_1 + \cdots + \varphi_n| = 2m\pi, \qquad m \in \{0, 1, 2, \ldots, k\}$$

and φ_j is given by (1.2). Then m is the *index* of $\mathcal{A}$, denoted as $\mathrm{Ind}(\mathcal{A})$.

For example, the heptagon on Fig. 1.2 has index equal to 1, since $|\varphi_1 + \cdots + \varphi_7| = 2\pi$. (See Figure 1.3. Let us remark that φ_4 is positively and all other angles are negatively oriented.)

Definition 1.4. A k-inscribed polygon $\mathcal{A}$ will be called a k-chordal polygon if it is of the first kind and $\mathrm{Ind}(\mathcal{A}) = k$.

Theorem A. *Let $\mathcal{A}$ be a k-chordal polygon and let β_j be given by (1.1). Then we have*

$$|\beta_1 + \cdots + \beta_n| = (n - 2k)\frac{\pi}{2}.$$

Proof. Since every k-chordal polygon is of the first kind (Definition 1.4), then either $\beta_j > 0,\ j = \overline{1,n}$ or $\beta_j < 0,\ j = \overline{1,n}$. If $\beta_j > 0$, then $\varphi_j < 0$ and the following holds

$$\varphi_1 + \cdots + \varphi_n = -2k\pi.$$

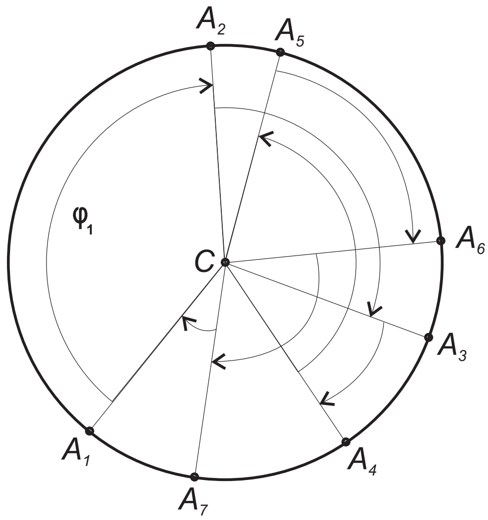

Figure 1.3:

In this case, because $2\beta_j + |\varphi_j| = \pi$ or $\varphi_j = 2\beta_j - \pi$, the above equality can be written as

$$\sum_{j=1}^{n} (2\beta_j - \pi) = -2k\pi,$$

or equivalently

$$\sum_{j=1}^{n} \beta_j = (n - 2k)\frac{\pi}{2}.$$

If $\beta_j < 0$, then $\varphi_j > 0$ and it holds $\varphi_1 + \cdots + \varphi_n = 2k\pi$. In this case we have

$$\sum_{j=1}^{n} \beta_j = -(n - 2k)\frac{\pi}{2}.$$

□

If $\mathcal{A}$ is a k-chordal polygon, then each $\beta_j,\ j = \overline{1,n}$, is negative if $\mathcal{A}$ is positively oriented and vice versa. But in the case when $\mathcal{A}$ is a k-inscribed polygon of the second kind, then some of the β_j are negative and some are positive.

Remark 1.4. In the sequel, for the sake of simplicity, we shall assume that the considered polygon is negatively oriented. Thus, in the case when a k-inscribed polygon $\mathcal{A}$ is negatively oriented, then

$$\varphi_1 + \cdots + \varphi_n \le 0 \quad \text{but} \quad \beta_1 + \cdots + \beta_n \ge 0.$$

Finally, let us point out that for $\text{Ind}(\mathcal{A}) = 0$, the following holds

$$\varphi_1 + \cdots + \varphi_n = 0 = \beta_1 + \cdots + \beta_n.$$

Theorem B. *Let $\mathcal{A}$ be a k-inscribed polygon. Then*

$$|\beta_1 + \cdots + \beta_n| = [n - 2(m + \nu)]\frac{\pi}{2}, \tag{1.3}$$

where Ind$(\mathcal{A}) = m$ and ν is number of all negative β_j's.

Proof. As $\varphi_j = -\pi + 2\beta_j$ if $\beta_j > 0$ and $\varphi_j = \pi + 2\beta_j$ if $\beta_j < 0$, the equality $\varphi_1 + \cdots + \varphi_n = -2m\pi$ can be written as

$$2\beta_1 + \cdots + 2\beta_n + \nu\pi - (n-\nu)\pi = -2m\pi,$$

from which (1.3) follows.

If $\beta_{j_1}, \dots, \beta_{j_\nu}$ are the negative angles in (1.3), then we have

$$|\beta_1| + \cdots + |\beta_n| = [n - 2(m+\nu)]\frac{\pi}{2} + 2\tau, \tag{1.4}$$

where $\tau = -(\beta_{j_1} + \cdots + \beta_{j_\nu})$. □

The greatest part of this article is in some way connected to the following theorem, see [5, Theorem 1] as well.

Theorem C. *Let $\mathcal{A}$ be a k-chordal polygon. Then*

$$\sum_{j=1}^{n} \cos\beta_j > 2k, \tag{1.5}$$

where

$$\sum_{j=1}^{n} \beta_j = (n-2k)\frac{\pi}{2}, \qquad 0 < \beta_j < \frac{\pi}{2}, j = \overline{1,n}.$$

Proof. Since $\cos\pi x > 1 - 2x$ if $x \in (0, 1/2)$, putting $\alpha = \pi x$ we obtain

$$\cos\alpha > 1 - \frac{2}{\pi}\,\alpha, \qquad 0 < \alpha < \frac{\pi}{2}. \tag{1.6}$$

Thus, we deduce

$$\sum_{j=1}^{n} \cos\beta_j > n - \frac{2}{\pi}\sum_{j=1}^{n}\beta_j = n - \frac{2}{\pi}(n-2k)\frac{\pi}{2} = 2k.$$

□

Remark 1.5. After this paper had been written, J. Sándor informed me that the inequality (1.6) follows from Jordan's inequality

$$\sin x > \frac{2}{\pi}\,x, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

putting $x = \pi/2 - \alpha$.

At this point let us remark that we can consult the articles [1], [2], [3], [4] and [8] for further information and generalizations of certain inequalities concerning plane and space polygons.

2. CERTAIN INEQUALITIES CONCERNING k-CHORDAL POLYGONS

In this section we deal with k-chordal polygons. By Remark 1.4 and Definition 1.4, all angles β_j are positive. First of all we give the following remark.

Remark 2.1. By the relation $\beta_j \approx 0$ we mean that β_j is near to zero, but it is positive. Similarly, $\beta_j \approx \pi/2$ denotes the case, when β_j is close to $\pi/2$, but it is less than $\pi/2$.

Theorem 2.2. *Let $k,\ n$ be positive integers such that $n - 2k > 0$ and let $\beta_1, \dots, \beta_n$ be angles such that*

$$\sum_{j=1}^{n} \beta_j = (n-2k)\frac{\pi}{2}, \qquad 0 < \beta_j < \frac{\pi}{2}, j = \overline{1,n}. \tag{2.1}$$

Then there exists a positive number h such that

$$\sum_{j=1}^{n} \cos^h \beta_j = 2k, \tag{2.2}$$

where

$$1 < h < \frac{\log \frac{2k}{2k+1}}{\log \cos \frac{\pi}{4k+2}}. \tag{2.3}$$

Proof. From (1.5) it follows that there is a positive h for which (2.2) holds as well. Now, we only need to prove that this h satisfies (2.3). For this purpose we will first prove the following lemma.

Lemma 2.3. *Let $h \geq 1$ be fixed. Then the function $y = \cos^h x$ is concave in the interval $(0, \arctan(1/\sqrt{h-1}))$.*

Proof. As

$$y'' = h \cos^{h-2} x \, [(h-1)\sin^2 x - \cos^2 x],$$

it follows that

$$\begin{aligned} y'' &< 0 \qquad \text{if} \quad (h-1)\tan^2 x < 1, \\ y'' &> 0 \qquad \text{if} \quad (h-1)\tan^2 x > 1. \end{aligned}$$

Thus, the function $y = \cos^h x$ is concave in $(0, \arctan(1/\sqrt{h-1}))$ and convex in the interval $(\arctan(1/\sqrt{h-1}), \pi/2)$. This proves Lemma 2.3. □

Now, assume that (2.1) is fulfilled. Then it is easy to see that the sum $\sum_{j=1}^{n} \cos^h \beta_j$ has the following properties.

(i_1) If $(n-2k)\frac{\pi}{2n} < \arctan(1/\sqrt{h-1})$, then the sum $\sum_{j=1}^{n} \cos^h \beta_j$ attains its maximum for $\beta_1 = \cdots = \beta_n = (n-2k)\frac{\pi}{2n}$.

(i_2) If $(n-2k)\frac{\pi}{2n} > \arctan(1/\sqrt{h-1})$, then the sum $\sum_{j=1}^{n} \cos^h \beta_j$ attains its minimum for $\beta_1 = \cdots = \beta_n = (n-2k)\frac{\pi}{2n}$.

(i_3) If $\beta_1 = \cdots = \beta_{2k} \approx 0,\ \beta_{2k+1} = \cdots = \beta_n \approx \frac{\pi}{2}$, then

$$\sum_{j=1}^{n} \cos^h \beta_j \approx 2k.$$

(i_4) For h sufficiently large the following result holds:

$$n \cos^h (n-2k)\frac{\pi}{2n} < 2k.$$

(i_5) There are $h_1 \geq 1,\ h_2 > 1$ such that

$$n \cos^{h_1} (n-2k)\frac{\pi}{2n} > 2k, \qquad n \cos^{h_2} (n-2k)\frac{\pi}{2n} < 2k,$$

and the equality $n \cos^{h_0} (n-2k)\frac{\pi}{2n} = 2k$ is obtained for

$$h_0 = h(n,k) = \frac{\log \frac{2k}{n}}{\log \cos (n-2k)\frac{\pi}{2n}}. \tag{2.4}$$

Lemma 2.4. *Let* $h(k),\ k \in \mathbb{N}$ *be given by*

$$h(k) = \frac{\log \frac{2k}{2k+1}}{\log \cos \frac{\pi}{4k+2}}. \tag{2.5}$$

Then the sum $\sum_{j=1}^{n} \cos^{h(k)} \beta_j$ *attains its maximum for*

$$\beta_1 = \cdots = \beta_{2k+1} \approx \frac{\pi}{4k+2}, \qquad \beta_{2k+2} = \cdots = \beta_n \approx \frac{\pi}{2}. \tag{2.6}$$

Proof. Firstly let us remark that $\frac{\pi}{4k+2} = \frac{\pi}{2} : (2k+1)$ and this practically means that

$$\beta_{2k+2} + \cdots + \beta_n = (n - (2k+1))\frac{\pi}{2},$$

so, from

$$(2k+1)\cos^{h(k)} \frac{\pi}{4k+2} + (n - (2k+1))\cos^{h(k)} \frac{\pi}{2} = 2k \tag{2.7}$$

we get (2.5). To prove Lemma 2.4 we have to prove the inequality

$$\arctan \frac{1}{\sqrt{h(k)-1}} > \frac{\pi}{4k+2}. \tag{2.8}$$

Starting from (2.6), we can write

$$\sqrt{h(k)-1} < \cot \frac{\pi}{4k+2},$$

i.e.

$$h(k) < 1 + \cot^2 \frac{\pi}{4k+2},$$

so

$$\frac{\log \frac{2k}{2k+1}}{\log \cos \frac{\pi}{4k+2}} < 1 + \cot^2 \frac{\pi}{4k+2},$$

implying

$$\log \frac{2k}{2k+1} > \log \left(\cos \frac{\pi}{4k+2} \right)^{1/\sin^2 \frac{\pi}{4k+2}},$$

thus

$$\frac{2k}{2k+1} > \frac{1}{\sqrt{\left(1 - \sin^2 \frac{\pi}{4k+2}\right)^{-1/\sin^2 \frac{\pi}{4k+2}}}}.$$

Letting $k \to \infty$ in the last relation we get a valid result since the expression on the left-hand side tends to 1, while the right-hand side tends to $1/\sqrt{e}$. This finishes the proof of Lemma 2.4. $\square$

Finally we have to show that

$$(2k+1)\cos^{h(k)} \frac{\pi}{4k+2} > 2k,$$
$$(2k+1)\cos^{h(k)} \frac{\pi}{4k+2} > (2k+2)\cos^{h(k)} \frac{\pi}{2k+2},$$

where one can write $2k = 2k\cos^{h(k)} 0$, $\frac{\pi}{2k+2} = (\frac{\pi}{2} + \frac{\pi}{2}) : (2k+2)$. For this purpose it is sufficient to check that the above relations hold, e.g. for $k = 1, 2, 3$. Thus, we have

$$3\cos^{h(1)}\frac{\pi}{6} = 2.000000001,$$
$$5\cos^{h(2)}\frac{\pi}{10} = 4,$$
$$7\cos^{h(3)}\frac{\pi}{14} = 6.00000006,$$

$$4\cos^{h(1)}\frac{\pi}{4} = 1.50585114 < 3\cos^{h(1)}\frac{\pi}{6},$$
$$6\cos^{h(2)}\frac{\pi}{6} = 3.164961846 < 5\cos^{h(2)}\frac{\pi}{10},$$
$$8\cos^{h(3)}\frac{\pi}{8} = 4.947027176 < 7\cos^{h(3)}\frac{\pi}{14}.$$

Let us remark that $\frac{\pi}{4k+2} \approx \frac{1}{2}\frac{\pi}{2k+2}$ for sufficiently large k. This completes the proof of the Theorem 2.2. □

As an interesting illustrative example we provide the following table.

k	$h(k)$	$\arctan 1/\sqrt{h(k)-1}$	$\frac{\pi}{4k+2}$
1	2.818841678	36.55639173^0	30^0
2	4.446703708	28.30865018^0	18^0
3	6.070896923	23.94487335^0	12.85714286^0
4	7.693796543	21.13214916^0	10^0
5	9.316082999	19.12497372^0	8.18181812^0
10	17.42431500	13.86082784^0	4.28571428^0
100	163.3293834	04.48781187^0	0.44776119^0

Table 1.

Example 2.1. We give an illustrative example with respect to $h(2)$. The function $y = \cos^{h(2)} x$ is shown in Fig. 2.1 for $x \in [0, \frac{\pi}{2}]$. The point $x_0 = \arctan 1/\sqrt{h(2)-1} = 28.30865018$ is its inflection point. For $n = 11$, under the constraint (2.1), the sum $\sum_{j=1}^{11} \cos^{h(2)} \beta_j$ takes its

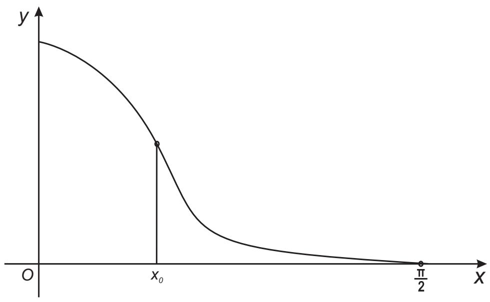

Figure 2.1:

maximum for

$$\beta_1 = \cdots = \beta_5 = \frac{\pi}{10}, \qquad \beta_6 = \cdots = \beta_{11} \approx \frac{\pi}{2}.$$

Here we point out that $y = \cos^{h(2)} x$ is concave in $(0, x_0)$ and

$$5\cos^{h(2)} \frac{\pi}{10} \geq \sum_{j=1}^{5} \cos^{h(2)} x_j,$$

holds true for every $x_1, \ldots, x_5$ such that $x_1 + \cdots + x_5 = \frac{\pi}{2}, \quad 0 < x_j < \frac{\pi}{2},\ j = \overline{1,5}$.

Also,

$$\begin{aligned}
5\cos^{h(2)} \frac{\pi}{10} > 6\cos^{h(2)} \frac{2\pi}{12} &= 3.164961846 > \\
> 7\cos^{h(2)} \frac{3\pi}{14} &= 2.343170592 > \\
> 8\cos^{h(2)} \frac{4\pi}{16} &= 1.713146048 > \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\
> 11\cos^{h(2)} \frac{7\pi}{22} &= 0.714031536,
\end{aligned}$$

holds, where

$$\begin{aligned}
&\frac{2\pi}{12} = \left(\frac{\pi}{2} + \frac{\pi}{2}\right) : 6, && \beta_7 = \cdots = \beta_{11} \approx \frac{\pi}{2}, \\
&\frac{3\pi}{14} = \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) : 7, && \beta_8 = \cdots = \beta_{11} \approx \frac{\pi}{2},
\end{aligned}$$

etc.

These relations can be clearly explained by the convexity of $\cos^{h(2)} x$ on $(x_0, \frac{\pi}{2})$ and by $x_0 < \frac{2\pi}{12} < \frac{3\pi}{14} < \cdots < \frac{7\pi}{22}$.

Now, we shall state and prove some corollaries of Theorem 2.2.

Corollary 2.5. *One has $h(k) \to \infty$ when $k \to \infty$.*

Proof. It can be found that

$$\frac{\frac{d}{dk}\left(\log \frac{2k}{2k+1}\right)}{\frac{d}{dk}\left(\log \cos \frac{\pi}{4k+2}\right)} = \frac{2k+1}{4} \cot \frac{\pi}{4k+2}.$$

□

For example, $h(500) = 811.78,\ h(10^3) = 1622.38,\ h(10^4) = 16233.22$, etc.

Corollary 2.6. *$h(k)$ is the same for all $n > 2k$.*

Proof. This is a consequence of (2.7). □

Corollary 2.7. *Let k be a fixed positive integer and $h(n,k)$ be given by (2.4). Then $h(n,k) \to 1$ when $n \to \infty$.*

Proof. It can be easily seen that

$$\frac{\frac{d}{dn}\left(\log \frac{2k}{n}\right)}{\frac{d}{dn}\left(\log \sin \frac{k\pi}{n}\right)} = \frac{n}{k\pi} \tan \frac{k\pi}{n}\ .$$

Now, obvious transformations give the assertion. □

For example, we have

$$h(5,1) = 1.72432, \quad h(6,1) = 1.58496, \quad h(7,1) = 1.50035,$$

$$h(5,2) = 4.44670, \quad h(6,2) = 2.81884, \quad h(7,2) = 2.27279.$$

Corollary 2.8. *Let $n_1, k_1,\ n_2, k_2$ be any given positive integers, such that $n_j > 2k_j,\ j = 1, 2$. If*

$$\frac{k_1}{n_1} = \frac{k_2}{n_2}, \tag{2.9}$$

then $h(n_1, k_1) = h(n_2, k_2)$.

Proof. Suppose that (2.9) holds. Then we can write

$$\frac{k_1\pi}{n_1} = \frac{k_2\pi}{n_2} \quad \Longrightarrow \quad (n_1 - 2k_1)\frac{\pi}{2n_1} = (n_2 - 2k_2)\frac{\pi}{2n_2}.$$

From this we easily deduce the assertion. □

Corollary 2.9. *Let $k \in \mathbb{N}$ be fixed. Then $h(n,k) \le h(k)$ for any integer $n > 2k$. The equality $h(n,k) = h(k)$ holds for $n = 2k + 1$.*

Proof. This follows from the Corollary 2.5 and Corollary 2.7. The asserted inequality is the straightforward consequence of (2.4) and (2.5). □

As an example we give the following numerical results (see Table 1 and the previous example):

$$h(5,1) = 1.72432 < h(1) = 2.81884$$
$$h(5,2) = 4.44670 = h(2) = 4.44670 \quad (\text{since } 5 = 2 \cdot 2 + 1)$$
$$h(6,2) = 2.81884 < h(2).$$

Theorem 2.10. *Let $\mathcal{A}$ be a given k-chordal n-gon and let $a_1, \dots, a_n$ be the lengths of its sides. Then*

$$\left(\frac{a_1^{h(k)} + \cdots + a_n^{h(k)}}{2k}\right)^{1/h(k)} \le 2r < \frac{a_1 + \cdots + a_n}{2k}, \tag{2.10}$$

where r denotes the radius of the circumcircle of $\mathcal{A}$.

Proof. From (2.2) and (2.3) it follows that

$$\sum_{j=1}^{n} \cos^{h(k)} \beta_j < 2k < \sum_{j=1}^{n} \cos \beta_j. \tag{2.11}$$

Since $a_j = 2r \cos \beta_j,\ j = \overline{1,n}$, the above inequalities can be written as in (2.10). Thus, Theorem 2.10 is proved. □

Corollary 2.11. *The following equality holds:*

$$a_1^{m(k)} + \cdots + a_n^{m(k)} = 2k(2r)^{m(k)}, \tag{2.12}$$

where $1 < m(k) \le h(k)$.

Corollary 2.12. *Let $a_1, \dots, a_n$ be given lengths. Then there exists a k-chordal n-gon with radius r whose sides have given lengths, if there is an $m(k)$ satisfying (2.12). (In this connection Example 2.6 may be interesting.)*

Corollary 2.13. *Let $a_1 = \cdots = a_n = a$. Then*

$$r = \frac{a}{2}\left(\frac{n}{2k}\right)^{1/h(n,k)}. \tag{2.13}$$

Proof. The relation (2.13) follows from (2.2) if $\beta_1 = \cdots = \beta_n$. □

Corollary 2.14. *The following equality holds:*

$$\sin\frac{k\pi}{n} = \left(\frac{2k}{n}\right)^{1/h(n,k)}.$$

Proof. As $a = 2r\cos(n-2k)\frac{\pi}{2n} = 2r\sin\frac{k\pi}{n}$, we have $a/(2r) = \sin\frac{k\pi}{n}$. From (2.13) it follows that

$$\frac{a}{2r} = \left(\frac{2k}{n}\right)^{1/h(n,k)}.$$

□

Example 2.2. Let $\beta_1 = 20^0$, $\beta_2 = 30^0$, $\beta_3 = 40^0$, $r = 5$. By the well-known relation $a_j = 2r\cos\beta_j$ we get

$$a_1 = 9.396926208, \quad a_2 = 8.660254038, \quad a_3 = 7.660444431.$$

From $\beta_1 + \beta_2 + \beta_3 = (3 - 2\cdot 1)\frac{\pi}{2}$, it is clear that $k = 1$. It can be found that

$$\cos^m\beta_1 + \cos^m\beta_2 + \cos^m\beta_3 = 1.999999783 \qquad \text{for} \qquad m = 2.737684,$$
$$\cos^m\beta_1 + \cos^m\beta_2 + \cos^m\beta_3 = 2.000000061 \qquad \text{for} \qquad m = 2.737683.$$

Thus, we have the approximative equality

$$a_1^m + a_2^m + a_3^m = 2k(2r)^m,$$

where $k = 1$ and $m = 2.737683$. We see immediately that $2.737683 < h(1) = 2.81884$. But it follows from the fact that β_j are not equal to each other, i.e. $\beta_j \neq \pi/6$. Therefore

$$\cos^{h(1)} 20^0 + \cos^{h(1)} 30^0 + \cos^{h(1)} 40^0 = 1.97761 < 2;$$

in the case of equal β_j's we have $3\cos^{h(1)}\pi/6 = 2$.

Example 2.3. Let $\beta_1 = 10^0$, $\beta_2 = 15^0$, $\beta_3 = 18^0$, $\beta_4 = 22^0$, $\beta_5 = 25^0, r = 4$. With the help of $a_j = 2r\cos\beta_j$ we derive

$$a_1 = 7.87846202,\ a_2 = 7.72740661,\ a_3 = 7.60845213,\ a_4 = 7.41747084,\ a_5 = 7.25046230.$$

From $\beta_1 + \cdots + \beta_5 = (5 - 2\cdot 2)\frac{\pi}{2}$ we conclude that $k = 2$. The corresponding pentagon is shown in Fig. 2.2. Let us remark that

$$\sum_{j=1}^{5} \text{measure of } \sphericalangle A_jCA_{j+1} = 4\pi.$$

It can be easily computed that

$$\sum_{j=1}^{5}\cos^m\beta_j = 3.999977021 \qquad \text{for} \qquad m = 4.2082782$$

$$\sum_{j=1}^{5}\cos^m\beta_j = 4.000022422 \qquad \text{for} \qquad m = 4.2082781.$$

Finally the approximate equality $\sum_{j=1}^{5} a_j^m = 2k(2r)^m$ holds for $k = 2$ and $m = 4.2082782$, where $m < h(2) = 4.446703708$.

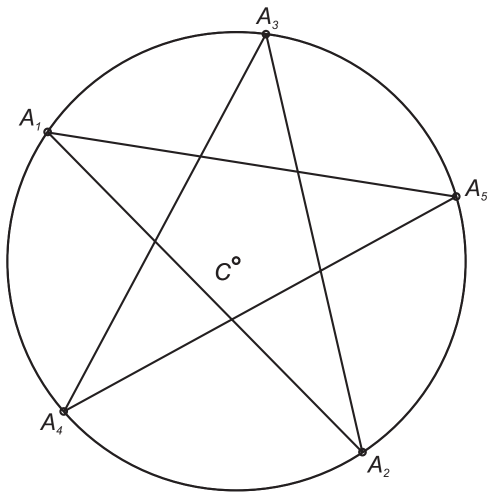

Figure 2.2:

Example 2.4. There is a 1-chordal pentagon $\mathcal{B}$ such that $b_j = |B_jB_{j+1}| = |A_jA_{j+1}| = a_j,\ j = \overline{1,5}$, where $\mathcal{A}$ is the 2-chordal pentagon shown in Fig. 2.2. It can be found that

$$\sum_{j=1}^{5} \arccos \frac{a_j}{12.90} = 270.011718^0 > 270^0,$$

$$\sum_{j=1}^{5} \arccos \frac{a_j}{12.89} = 269.955703^0 < 270^0.$$

Thus, the radius of the circumcircle of $\mathcal{B}$ satisfies the relation

$$12.89 < 2r_{\mathcal{B}} < 12.90$$

and for the angles of $\mathcal{B}$ we have $\beta_1 + \cdots + \beta_5 = (5 - 2 \cdot 1)\frac{\pi}{2}$, since, here $k = 1$.

Thus, besides the equality in Example 2.3 there is the equality

$$a_1^m + \cdots + a_5^m = 2k(2r_{\mathcal{B}})^m,$$

for $k = 1$ and $m < h(1) = 2.81884$.

Example 2.5. Let $\beta_1 = 9^0,\ \beta_2 = 63^0,\ \beta_3 = 65^0,\ \beta_4 = 66^0,\ \beta_5 = 67^0$, $r = 3$. Then there is 1-chordal pentagon such that

$$a_1^m + \cdots + a_5^m = 2 \cdot 6^m, \qquad 1 < m < h(1).$$

But there is no 2-chordal pentagon $\mathcal{B} \equiv B_1 \cdots B_5$ such that $a_j = |B_jB_{j+1}|$. Indeed, it is easy to show this by

$$a_1^m + \cdots + a_5^m < 4(2r_{\mathcal{B}})^m$$

for all $m \geq 1$, and for all $r_{\mathcal{B}} \geq 3\cos\beta_1$ when

$$a_1 = 5.92613,\ a_2 = 2.72394,\ a_3 = 2.53571,\ a_4 = 2.44042,\ a_5 = 2.34439\ .$$

Finally, we can show that for $m = 1$ and $m = h(2)$ we have

$$a_1^m + \cdots + a_5^m < 4a_1^m.$$

Definition 2.1. Let $\mathcal{A}$ be a k-chordal n-gon. Then the number $m > 1$ for which we obtain

$$\sum_{j=1}^{n} a_j^m = 2k(2r)^m$$

is the *characteristic* of $\mathcal{A}$ in the notation $\mathrm{Char}(\mathcal{A})$. Here r is the radius of the circumcircle of $\mathcal{A}$ and $a_j = |A_j A_{j+1}|,\ j = \overline{1,n}$.

Remark 2.15. By Theorem 2.2, we have

$$1 < \mathrm{Char}(\mathcal{A}) < h(k), \tag{2.14}$$

where $h(k)$ is given by (2.5).

In particular, if $\beta_1 = \cdots = \beta_n = (n-2k)\frac{\pi}{2n}$, then

$$\mathrm{Char}(\mathcal{A}) = h(n,k),$$

where $h(n,k)$ is given by (2.4).

Remark 2.16. Since there are situations when certain angles β_j are close to 0, and other angles are close to $\pi/2$, it is clear that instead of constraint (2.6) in the proving procedure of Theorem 2.2, we can take

$$\beta_1 = \cdots = \beta_{2k+1} = \frac{\pi}{4k+2}, \qquad \beta_{2k+2} = \cdots = \beta_n = \frac{\pi}{2}\ .$$

Thus, instead of (2.14) it can be written

$$1 < \mathrm{Char}(\mathcal{A}) \lesssim h(k),$$

where Char$(\mathcal{A}) \lesssim h(k)$ means that Char$(\mathcal{A}) < h(k)$ and that Char$(\mathcal{A})$ may be close to $h(k)$.

For example, let $\mathcal{A}$ be a 1-chordal pentagon such that

$$\beta_1 = \beta_2 = \beta_3 = \frac{90.000002^0}{3}, \quad \beta_4 = \beta_5 = 89.999999^0.$$

Then

$$\mathrm{Char}(\mathcal{A}) \approx h(1) = 2.818841678,$$

because

$$3\cos^{h(1)} \frac{90.000002^0}{3} + 2\cos^{h(1)} 89.999999^0 = 1.999999962 \approx 2.$$

Example 2.6. Let $\mathcal{A}$ be 1-chordal quadrilateral such that

$$\beta_1 + \beta_3 = \frac{\pi}{2} = \beta_2 + \beta_4.$$

Then $\mathrm{Char}(\mathcal{A}) = 2$. This is clear, since

$$\cos^2\beta_1 + \cos^2\beta_3 = 1 = \cos^2\beta_2 + \cos^2\beta_4.$$

Thus

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 2(2r)^2.$$

Of course this property is not true for every 1-chordal quadrilateral; there are chordal quadrilaterals where $\beta_1 + \beta_3 \neq \frac{\pi}{2}$, compare Fig 2.3.

Example 2.7. Let $\mathcal{A}$ be a 2-chordal octagon such that

$$\beta_1 + \beta_5 = \beta_2 + \beta_6 = \beta_3 + \beta_7 = \beta_4 + \beta_8 = \frac{\pi}{2}\ . \tag{2.15}$$

As an illustration, see Fig. 2.4

As

$$\cos^2\beta_j + \cos^2\beta_{j+4} = 1, \qquad j = 1,2,3,4,$$

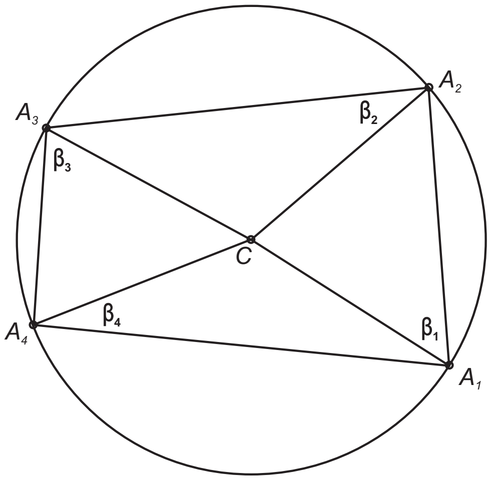

Figure 2.3:

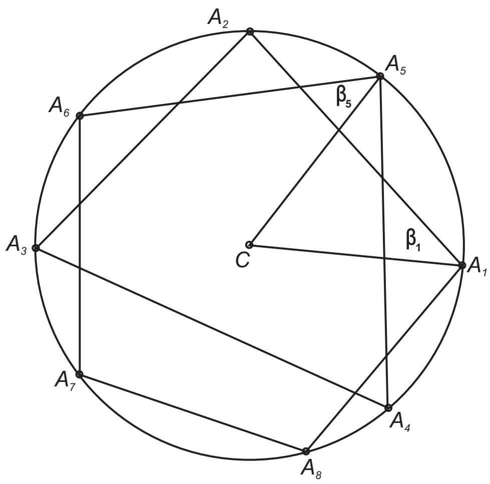

Figure 2.4:

then $\mathrm{Char}(\mathcal{A}) = 2$. Thus we clearly deduce by $\sum_{j=1}^{8} \cos^2 \beta_j = 4$ that

$$\sum_{j=1}^{8} a_j^2 = 4(2r)^2.$$

Of course, instead of (2.15) we can assume that

$$\beta_{i_1} + \beta_{i_2} = \beta_{i_3} + \beta_{i_4} = \beta_{i_5} + \beta_{i_6} = \beta_{i_7} + \beta_{i_8} = \frac{\pi}{2} \ .$$

Here $i_j \in \{1, 2, \ldots, 8\}$.

Theorem 2.17. *Let $\mathcal{A}$ be a k-chordal n-gon, where $n = 4k$, and let*

$$\beta_{i_1} + \beta_{i_2} = \cdots = \beta_{i_{n-1}} + \beta_{i_n} = \frac{\pi}{2}.$$

Then $\mathrm{Char}(\mathcal{A}) = 2$*.*

Proof. Since $\frac{4k}{2} = 2k$, we have

$$\sum_{j=1}^{n} \cos^2 \beta_{i_j} = 2k,$$

and this proves Theorem 2.17. □

Corollary 2.18. *We have*

$$\sum_{j=1}^{n} a_j^2 = 2k(2r)^2.$$

Theorem 2.19. *Let $\mathcal{A}$ be a chordal n-gon such that*

$$\sum_{j=1}^{n-1} \beta_j = (n-2)\frac{\pi}{2}, \quad \beta_n = 0, \quad 0 < \beta_j < \frac{\pi}{2}, \quad j = \overline{1, n-1}. \tag{2.16}$$

Then $\mathrm{Char}(\mathcal{A}) \leq 2$.

Proof. As it will be seen, this theorem is a corollary of Theorem 2.2. First we point out that (2.16) is obtained by putting $k = 1,\ \beta_n = 0$ into (2.1). Also, let us remark that in (2.1) we can take $\beta_n \approx 0$ as well. Therefore the proof of Theorem 2.19 is a straightforward consequence of Theorem 2.2 where, instead of (2.6), we write

$$\beta_1 = \cdots = \beta_{2k} \approx \frac{\pi}{4k}, \quad \beta_{2k+1} = \cdots = \beta_{n-1} \approx \frac{\pi}{2}; \quad \beta_n = 0,$$

or, because $k = 1$, we put

$$\beta_1 = \beta_2 \approx \frac{\pi}{4}, \quad , \beta_3 = \cdots = \beta_{n-1} \approx \frac{\pi}{2}, \quad \beta_n = 0.$$

For these specified values of $\beta_1, \ldots, \beta_n$ we obtain

$$\sum_{j=1}^{n} \cos^2 \beta_j \approx \cos^2 0 + 2\cos^2 \frac{\pi}{4} = 2,$$

and

$$\sum_{j=1}^{n} \cos^m \beta_j \approx \cos^m 0 + 2\cos^m \frac{\pi}{4} < 2$$

when $m > 2$. Theorem 2.19 is thus proved. □

Corollary 2.20. *Let the situation be the same as in Theorem 2.19. Then there is a m such that*

$$a_1^m + \cdots + a_{n-1}^m = a_n^m, \qquad 1 < m \leq 2.$$

Proof. The assertion immediately follows from $\sum_{j=1}^{n} a_j^m = 2(2r)^m$ because $a_n = 2r$. □

Corollary 2.21. *Under conditions of Theorem 2.19, $Char(\mathcal{A}) = 2$ only if $n = 3$.*

Proof. Without loss of generality we can take $n = 5$ and consider the pentagon in Fig. 2.5. By Theorem 2.19, we have $\sum_{j=1}^{5} \cos^2 \beta_j \leq 2$. But if $A_3 \to A_5$ and $A_4 \to A_5$, then

$$\beta_1 + \beta_2 \to \frac{\pi}{2}, \quad \beta_3, \beta_4 \to \frac{\pi}{2}, \quad \beta_5 = 0$$

and $\sum_{j=1}^{5} \cos^2 \beta_j \to 2$. □

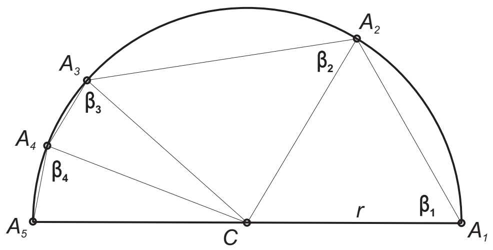

Figure 2.5:

Example 2.8. Specify $\beta_1 = 62^0, \beta_2 = 65^0, \beta_3 = 68^0, \beta_4 = 75^0$, then

$$\sum_{j=1}^{5} \cos^{3/2} \beta_j = 1.957416 < 2,$$

$$\sum_{j=1}^{5} \cos^{7/5} \beta_j = 2.050053 > 2.$$

When $\beta_1 = 44.1^0, \beta_2 = 46.9^0, \beta_3 = 89.4^0, \beta_4 = 89.6^0$, then

$$\sum_{j=1}^{5} \cos^2 \beta_j = 1.9827268 < 2,$$

$$\sum_{j=1}^{5} \cos^{19/10} \beta_j = 2.0183067 > 2.$$

Remark 2.22. As we can see, Theorem 2.19 may be considered as a generalization of the Pythagorean theorem. For example, all positive solutions of the equation

$$x_1^{3/2} + \cdots + x_{n-1}^{3/2} = x_n^{3/2}$$

are related to chordal n-gons whose characteristic is $3/2$.

Thus, the problem "find all positive solutions of the above equation" is in fact the problem "find all angles $\beta_1, \ldots, \beta_n$ such that (2.16) is satisfied under the constraint

$$\sum_{j=1}^{n} \cos^{3/2} \beta_j = 2.\text{''}$$

This problem is obvious when $n = 3$ since then $\beta_1 + \beta_2 = \frac{\pi}{2}$. But the case $n > 3$ could be very difficult.

Theorem 2.23. *Let $\mathcal{A}$ be a k-chordal n-gon. Then for every real $p > 1$, we have*

$$\sum_{j=1}^{n} \cos^p \beta_j > n \left(\frac{2k}{n} \right)^p, \tag{2.17}$$

where $\beta_1, \ldots, \beta_n$ satisfy (2.1).

Proof. In [6, Theorem 2] it was proved that (2.17) holds for every positive integer p. Here we give an abbreviated and simplified proof of this result by which we deduce the assertion of Theorem 2.23.

By the Jordan-type inequality (1.6), i.e. by

$$\cos\beta_j > 1 - \frac{2}{\pi}\beta_j,$$

using the properties of the arithmetical mean, we can write

$$\sum_{j=1}^{n} \cos^p \beta_j \geq n\left(\frac{1}{n}\sum_{j=1}^{n}\cos\beta_j\right)^p > n\left(\frac{1}{n}\sum_{j=1}^{n}\left(1-\frac{2}{\pi}\beta_j\right)\right)^p = n\left(\frac{2k}{n}\right)^p.$$

Indeed, here we have

$$\sum_{j=1}^{n}\left(1-\frac{2}{\pi}\beta_j\right) = n - \frac{2}{\pi}\sum_{j=1}^{n}\beta_j = n - \frac{2}{\pi}\,(n-2k)\frac{\pi}{2} = 2k.$$

This completes the proof of Theorem 2.23. □

Corollary 2.24. *Under the same assumptions as in Theorem 2.23, the following holds*

$$a_1^p + \cdots + a_n^p > n\left(\frac{4kr}{n}\right)^p.$$

For $p = 1$ one obtains an interesting relation:

$$a_1 + \cdots + a_n > 4kr.$$

For example if $n = 7,\ k = 3$, then $\sum_{j=1}^{7} a_j > 12r$.

3. INEQUALITIES CONCERNING k-INSCRIBED POLYGONS

In this section we start with the equality (1.4):

$$|\beta_1| + \cdots + |\beta_n| = [n - 2(m+\nu)]\frac{\pi}{2} + 2\tau \tag{1.4}$$

where $\tau = -(\beta_{j_1} + \cdots + \beta_{j_\nu})$ and $\beta_{j_1}, \ldots, \beta_{j_\nu}$ are the negative angles, while $\operatorname{Ind}(\mathcal{A}) = m$.

Let λ be defined by $2\tau = \lambda\pi$. Then (1.4) becomes

$$|\beta_1| + \cdots + |\beta_n| = [n - 2(m+\nu-\lambda)]\frac{\pi}{2}. \tag{3.1}$$

Using the inequality (1.6) we can write

$$\sum_{j=1}^{n}\cos\beta_j > \sum_{j=1}^{n}\left(1 - \frac{2}{\pi}|\beta_j|\right) = n - \frac{2}{\pi}\sum_{j=1}^{n}|\beta_j| = 2(m+\nu-\lambda).$$

So, by (3.1) it follows that

$$\sum_{j=1}^{n}\cos\beta_j > 2(m+\nu-\lambda). \tag{3.2}$$

In the case $\operatorname{Ind}(\mathcal{A}) = k,\ \nu = 0$, we get the inequality (1.5).

It can be easily seen that

$$\nu - \lambda > 0. \tag{3.3}$$

Let us remark that

$$2\tau = 2\sum_{j=1}^{\nu}|\beta_{i_j}| < 2\left(\nu\cdot\frac{\pi}{2}\right),$$

from which it follows that $2\tau < \nu\pi$.

For the sake of brevity, we denote in the sequel

$$w = \operatorname{Ind}(\mathcal{A}) + \nu - \lambda. \tag{3.4}$$

Now, we have the following theorem which is in fact a corollary of Theorem 2.2.

Theorem 3.1. *Let $\mathcal{A}$ be a k-inscribed n-gon and let*

$$\sum_{j=1}^{n} |\beta_j| = (n-2w)\frac{\pi}{2}, \qquad 0 < |\beta_j| < \frac{\pi}{2}. \tag{3.5}$$

Then there is a q such that

$$\sum_{j=1}^{n} \cos^q \beta_j = 2w, \qquad 1 < q \le h(w), \tag{3.6}$$

where

$$h(w) = \frac{\log \frac{2w}{2w+1}}{\log \cos \frac{\pi}{4w+2}} \quad \textit{if } w \textit{ is an integer,} \tag{3.7}$$

but if w is not an integer, that is, when $\lambda = z + u$, where $z \ge 0$ is an integer and u is a positive number such that $0 < u < 1$, then

$$h(w) = \frac{\log \frac{\operatorname{Ind}(\mathcal{A})+\nu-\lambda}{\operatorname{Ind}(\mathcal{A})+\nu-z}}{\log \cos \frac{u\pi}{2(\operatorname{Ind}(\mathcal{A})+v-z)}}. \tag{3.8}$$

Proof. If w is an integer then the proof is quite analogous to the proof of Theorem 2.2.

In the case when w is not an integer, then instead of (2.6), we have the expressions

$$|\beta_1| = \cdots = |\beta_{2(\operatorname{Ind}(\mathcal{A})+\nu-z)}| \approx \frac{u\pi}{2(\operatorname{Ind}(\mathcal{A})+\nu-z)}, \quad |\beta_{2(\operatorname{Ind}(\mathcal{A})+\nu-z)+1}| = \cdots = |\beta_n| \approx \frac{\pi}{2}.$$

Let us remark that now the equality from (3.5) can be written as

$$\sum_{j=1}^{n} |\beta_j| = [n - 2\left(\operatorname{Ind}(\mathcal{A}) + \nu - z\right)]\frac{\pi}{2} + u\pi$$

since

$$[n - 2(\operatorname{Ind}(\mathcal{A}) + \nu - z - u)]\frac{\pi}{2} = [n - 2(\operatorname{Ind}(\mathcal{A}) + \nu - z)]\frac{\pi}{2} + u\pi$$

Thus, from

$$2(\operatorname{Ind}(\mathcal{A}) + \nu - z)\cos^{h(w)} \frac{u\pi}{2(\operatorname{Ind}(\mathcal{A}) + \nu - z)} = 2w$$

we get (3.8).

Theorem 3.1 is thus proved. □

Corollary 3.2. *Let $a_i = |A_iA_{i+1}|,\ i = 1, \ldots, n$. Then*

$$\sum_{j=1}^{n} a_j^m = 2w(2r)^m, \qquad 1 < m \lesssim h(w).$$

Example 3.1. In Fig. 3.1 we have drawn an 1-inscribed pentagon $\mathcal{A}$, with $r = 3$ and $\beta_1 = -6^0, \beta_2 = 56^0, \beta_3 = 28^0, \beta_4 = -58^0, \beta_5 = 70^0$. In this case we have $k = 1, \operatorname{Ind}(\mathcal{A}) = 0, \nu = 2$ and

$$\sum_{j=1}^{5} \beta_j = 90^0 = \frac{\pi}{2} \text{ radian}, \qquad \sum_{j=1}^{5} |\beta_j| = 218^0 = \left(\frac{\pi}{2} + 2\cdot\frac{64}{90}\cdot\frac{\pi}{2}\right) \text{ radian}.$$

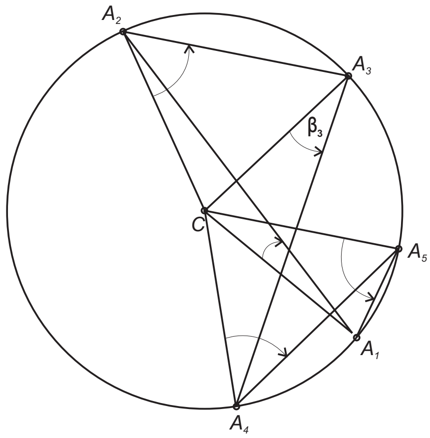

Figure 3.1:

Thus we can write

$$\sum_{j=1}^{5} |\beta_j| = (5-2\nu)\frac{\pi}{2} + 2\cdot\frac{64}{90}\cdot\frac{\pi}{2} = (5-2(\nu-0.71111111))\frac{\pi}{2}.$$

Since $\nu = 2$, we have

$$w = \nu - 0.71111111 = 1.28888888,\ 2w = 2.57777776$$
$$\lambda = 0.71111111,\ z = 0,\ u = \lambda \text{ since } \lambda < 1.$$

Finally, it can be written that

$$\sum_{j=1}^{5} \cos^m \beta_j = 2.571645882 < 2w \qquad \text{for } m = 1.85,$$
$$\sum_{j=1}^{5} \cos^m \beta_j = 2.578128456 > 2w \qquad \text{for } m = 1.84.$$

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