

ON A GENERALIZATION OF LIPSCHITZ'S CLASSES

Xh. Z. KRASNIQI

Department of Mathematics and Computer Sciences,
Avenue "Mother Teresa" 5
Prishtinë, 10000, Republic of Kosovo
EMail: xheki00@hotmail.com

Received: 26 March, 2008

Accepted: 06 August, 2008

Communicated by: H. Bor

2000 AMS Sub. Class.: 42A20, 42A32.

Key words: Lipschitz classes, Fourier series.

Abstract: In this paper we obtain a generalization of Lipschitz's classes $\Lambda^m(\beta, p, r)$ defined in [1]. We give necessary conditions for even or odd functions with Fourier series to belong to the classes $\Lambda^m(p, r, \alpha)$. We also give sufficient conditions for even or odd functions with Fourier series to belong to the same classes.



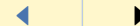
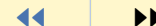
Generalization of Lipschitz's
Classes

Xh. Z. Krasniqi

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1. Definitions and Useful Statements

We consider the series

$$(1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

or

$$(1.2) \quad \sum_{n=1}^{\infty} a_n \sin nx$$

where a_n are Fourier coefficients of integrable function f .

Definition 1.1. We say that a function f belongs to WA^p , ($1 < p < \infty$) if

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p < +\infty$$

where $\Delta a_k = a_k - a_{k+1}$ (see [1]).

We say that any function $\alpha(t)$ is a function of type σ (see [4]) if it is measurable in $[0, 1]$, integrable in $[\delta, 1]$ for each $\delta \in (0, 1)$, and there exist real numbers $C_{1,\alpha} > 0$, σ and $\delta_0 \in (0, 1)$ such that

1. $\alpha(t) \geq C_{1,\alpha}$, for all $t \in [0, 1]$;
2. $\int_0^\delta \alpha(t)t^s dt < \infty$ for each $s > \sigma$ and $\delta \in (0, \delta_0)$;
3. $\int_0^\delta \alpha(t)t^s dt = \infty$ for each $s < \sigma$ and $\delta \in (0, \delta_0)$, and

$$\int_0^\delta \alpha(t)t^\sigma dt \leq C_2 \delta^\sigma \int_\delta^{2\delta} \alpha(t)dt.$$

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In [1], the classes $\Lambda^m(\beta, p, r,)$ are defined in the following way

Definition 1.2. $f \in \Lambda^m(\beta, p, r,)$, if

$$\|f\|_{\beta,p,r}^{(m)} \equiv \left\{ \int_0^1 \left[\int_0^{2\pi} \frac{|\Delta_m f(x,t)|^p}{t^{\beta p}} dx \right]^{\frac{r}{p}} dt \right\}^{\frac{1}{r}} < +\infty,$$

where $1 < p < +\infty, 1 \leq r < +\infty, \beta > 0, m \in \mathbb{N}$ and

$$\Delta_m f(x,t) = \sum_{i=1}^m (-1)^i C_m^i f[x + (m - 2i)t].$$

Now we define classes $\Lambda^m(p, r, \alpha)$ as follows:

Definition 1.3. We say $f \in \Lambda^m(p, r, \alpha)$, if

$$\|f\|_{p,r,\alpha}^{(m)} \equiv \left\{ \int_0^1 \alpha(t) \left[\int_0^{2\pi} |\Delta_m f(x,t)|^p dx \right]^{\frac{r}{p}} dt \right\}^{\frac{1}{r}} < +\infty,$$

where $\alpha(t)$ is function of the type σ .

For $\alpha(t) = t^{-r\beta-1}, \beta > 0$, we get the classes $\Lambda^m(\beta, p, r)$, considered in [1]. Therefore classes $\Lambda^m(p, r, \alpha)$ are generalizations of classes $\Lambda^m(\beta, p, r)$.

We need some auxiliary statements.



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Lemma 1.4 ([2]). Let a_ν , b_ν and β_n be numbers such that $a_\nu \geq 0$, $b_\nu \geq 0$ and $\sum_{\nu=n}^{\infty} a_\nu = a_n \beta_n$:

1. For $0 < p \leq 1$ the following inequality is valid

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=1}^{\nu} b_\mu \right)^p \geq p^p \sum_{\nu=1}^{\infty} a_\nu (b_\nu \beta_\nu)^p;$$

2. For $1 \leq p < \infty$ we have

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=1}^{\nu} b_\mu \right)^p \leq p^p \sum_{\nu=1}^{\infty} a_\nu (b_\nu \beta_\nu)^p.$$

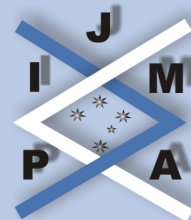
Lemma 1.5 ([2]). Let a_ν , b_ν and γ_n be numbers such that $a_\nu \geq 0$, $b_\nu \geq 0$ and $\sum_{\nu=1}^n a_\nu = b_n \gamma_n$:

1. For $0 < p \leq 1$, we have

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=\nu}^{\infty} b_\mu \right)^p \geq p^p \sum_{\nu=1}^{\infty} a_\nu (b_\nu \gamma_\nu)^p;$$

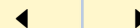
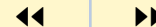
2. For $1 \leq p < \infty$, we have

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=\nu}^{\infty} b_\mu \right)^p \leq p^p \sum_{\nu=1}^{\infty} a_\nu (b_\nu \gamma_\nu)^p.$$



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Lemma 1.6 ([3]). Let μ , τ and a_ν be numbers such that $0 < \mu < \tau < \infty$ and $a_\nu \geq 0$. Then

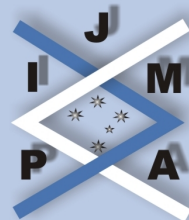
$$\left(\sum_{\nu=1}^{\infty} a_\nu^\tau \right)^{\frac{1}{\tau}} \leq \left(\sum_{\nu=1}^{\infty} a_\nu^\mu \right)^{\frac{1}{\mu}}.$$

We denote by C a constant that depends only on m, p, r and may be different in different relations.

Theorem 1.7 ([1]). If $f \in WA^p$, $1 < p < +\infty$, then

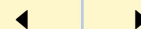
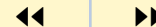
$$\{\omega_p^{(m)}(h; f)\}^p \leq Ch^{mp} \sum_{n \leq [\frac{1}{h}]} n^{(m+1)p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p + C \sum_{n > [\frac{1}{h}]} n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p,$$

where $\omega_p^{(m)}(h; f)$ is the integral modulus of smoothness of order m .



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2. Main Results

Let us denote

$$A(n) := \int_{1/(n+1)}^{1/n} \alpha(t) dt,$$

$$b(n) := b_1(n) + b_2(n) = n^{mr} \int_0^{1/n} \alpha(t) t^{mr} dt + \int_{1/(n+1)}^1 \alpha(t) dt.$$

We have the following first main result.

Theorem 2.1. *Let m be any natural number and*

$$f \in AW^p, \quad 1 < p < +\infty, \quad 1 \leq r < +\infty.$$

If for the coefficients of series (1.1) or (1.2) we have $\sum_{k=1}^{\infty} |\Delta a_k| < +\infty$, then:

1. *For $p \leq r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \left(\frac{b(n)}{A(n)} \right)^{\frac{r}{p}-1} \right\}^{\frac{1}{r}};$$

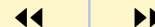
2. *For $p > r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}}.$$



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Proof. Using the characteristics of the integral modulus of smoothness we have

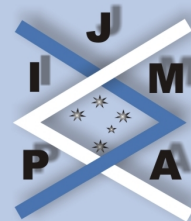
$$\begin{aligned} \left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r &= \int_0^1 \alpha(t) \left[\int_0^{2\pi} |\Delta_m f(x,t)|^p dx \right]^{\frac{r}{p}} dt \\ &\leq \sum_{N=1}^{\infty} \int_{1/(N+1)}^{1/N} \alpha(t) [\omega_p^{(m)}(f;t)]^r dt \\ &\leq \sum_{N=1}^{\infty} [\omega_p^{(m)}(f;1/N)]^r \int_{1/(N+1)}^{1/N} \alpha(t) dt \\ &= \sum_{N=1}^{\infty} A(N) [\omega_p^{(m)}(f;1/N)]^r. \end{aligned}$$

According to the Theorem 1.7, we have

$$\begin{aligned} \left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r &\leq C \sum_{N=1}^{\infty} A(N) N^{-mr} \left\{ \sum_{n=1}^N n^{(m+1)p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p \right\}^{\frac{r}{p}} \\ &\quad + C \sum_{N=1}^{\infty} A(N) \left\{ \sum_{n=N+1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p \right\}^{\frac{r}{p}} \\ &= I_1 + I_2. \end{aligned}$$

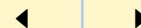
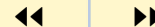
Now we estimate I_1 and I_2 . Let $r/p \geq 1$. Then according to Lemma 1.4 we have

$$I_1 \leq C \sum_{N=1}^{\infty} A(N) N^{-mr} \left\{ N^{(m+1)p-2} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^p \beta_N \right\}^{\frac{r}{p}}.$$



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Now we estimate the quantity β_N :

$$\begin{aligned} A(N)N^{-mr}\beta_N &= \sum_{i=N}^{\infty} A(i)i^{-mr} \\ &= \sum_{i=N}^{\infty} \left(\frac{i+1}{i}\right)^{mr} \cdot \frac{1}{(i+1)^{mr}} \int_{1/(i+1)}^{1/i} \alpha(t)dt \\ &\leq 2^{mr} \int_0^{1/N} \alpha(t)t^{mr} dt, \end{aligned}$$

or

$$\beta_N \leq C \frac{b_1(N)}{A(N)}.$$

Consequently

$$(2.1) \quad I_1 \leq C \sum_{N=1}^{\infty} A(N)N^{r(1-\frac{2}{p})} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^r \left\{ \frac{b_1(N)}{A(N)} \right\}^{\frac{r}{p}}.$$

According to Lemma 1.5, for $r/p \geq 1$ we have

$$I_2 \leq C \sum_{N=1}^{\infty} A(N) \left\{ N^{p-2} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^p \gamma_N \right\}^{\frac{r}{p}}.$$

We estimate the quantity γ_N :

$$A(N)\gamma_N = \sum_{i=1}^N A(i) = \int_{1/(N+1)}^1 \alpha(t)dt \Rightarrow \gamma_N = \frac{b_2(N)}{A(N)}.$$



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Consequently

$$(2.2) \quad I_2 \leq C \sum_{N=1}^{\infty} A(N) N^{r(1-\frac{2}{p})} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^r \left\{ \frac{b_2(N)}{A(N)} \right\}^{\frac{r}{p}}.$$

By (2.1) and (2.2) we get

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \leq C \sum_{N=1}^{\infty} A(N) N^{r(1-\frac{2}{p})} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^r \left\{ \left(\frac{b_1(N)}{A(N)} \right)^{\frac{r}{p}} + \left(\frac{b_2(N)}{A(N)} \right)^{\frac{r}{p}} \right\}.$$

Finally, according to Lemma 1.6, for $r \geq p$ we have

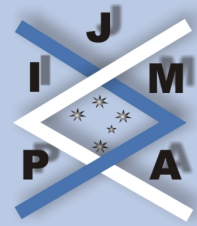
$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{N=1}^{\infty} A(N) N^{r(1-\frac{2}{p})} \left(\sum_{k=N}^{\infty} |\Delta a_k| \right)^r \left(\frac{b(N)}{A(N)} \right)^{\frac{r}{p}} \right\}^{\frac{1}{r}}.$$

Now let $r/p < 1$. Then, according to Lemma 1.6, we have

$$I_1 \leq C \sum_{N=1}^{\infty} A(N) N^{-mr} \sum_{n=1}^N n^{(m+1)r-\frac{2r}{p}} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r.$$

If we change the order of summation we get

$$(2.3) \quad \begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} n^{(m+1)r-\frac{2r}{p}} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \sum_{N=n}^{\infty} A(N) N^{-mr} \\ &\leq C \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b_1(n). \end{aligned}$$



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Now we estimate I_2 . Using Lemma 1.6 and changing the order of summation we have:

$$\begin{aligned}
 I_2 &\leq C \sum_{N=1}^{\infty} A(N) \sum_{n=N}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \\
 &= C \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \sum_{N=1}^n A(N) \\
 (2.4) \quad &= C \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b_2(n).
 \end{aligned}$$

From (2.3) and (2.4) we deduce

$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}},$$

which fully demonstrates Theorem 2.1. □

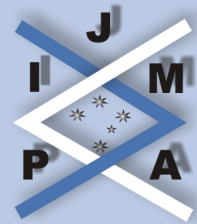
Theorem 2.2. Let m be any natural number and

$$1 < p \leq 2, \quad 1 \leq r < +\infty, \quad 1/p + 1/q = 1.$$

If a_n are the coefficients of series (1.1) or (1.2), then:

1. For $r \leq q$ we have

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \left[\frac{b_1(n)}{A(n)} \right]^{\frac{r-1}{q}} \right\}^{\frac{1}{r}};$$



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2. For $r > q$ we have

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \right\}^{\frac{1}{r}}.$$

Proof. Let f be an even function. If f is an odd function then the proof of the theorem is analogous to the even case. It is not difficult to see that the Fourier series of $\Delta_m f(x, t)$ is

$$\Delta_m f(x, t) \sim \begin{cases} (-1)^{\frac{m}{2}} 2^m \sum_{n=1}^{\infty} a_n \cos nx \sin^m nt, & \text{for } m \text{ even} \\ (-1)^{\frac{m-1}{2}-1} 2^m \sum_{n=1}^{\infty} a_n \sin nx \sin^m nt, & \text{for } m \text{ odd.} \end{cases}$$

According to the well-known Hausdorff-Young's theorem we find

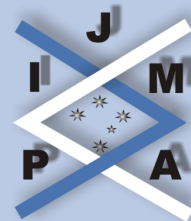
$$C \left(\int_0^{2\pi} |\Delta_m f(x, t)|^p dx \right)^{\frac{r}{p}} \geq \left(\sum_{n=1}^{\infty} |a_n|^q |\sin nt|^{mq} \right)^{\frac{r}{q}},$$

and then

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} \int_{1/(\nu+1)}^{1/\nu} \alpha(t) \left(\sum_{n=1}^{\nu} |a_n|^q |\sin nt|^{mq} \right)^{\frac{r}{q}} dt.$$

Using the well-known inequality $\sin B \geq \frac{2}{\pi} B$ for $0 \leq B \leq \frac{\pi}{2}$, we get

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} A(\nu) \left(\sum_{n=1}^{\nu} n^{-mq} |a_n|^q \right)^{\frac{r}{q}}.$$



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Let $r \leq q$, then according to Lemma 1.4 we have

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} A(\nu) [\nu^{-mq} |a_\nu|^q \beta_\nu]^{\frac{r}{q}}.$$

It is easy to prove that $\beta_\nu \geq \frac{b_1(\nu)}{A(\nu)}$, from which we get

$$(2.5) \quad \left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} \nu^{-mr} |a_\nu|^r b_1(\nu) \left[\frac{b_1(\nu)}{A(\nu)} \right]^{\frac{r}{q}-1}.$$

Let $q < r$, then according to Lemma 1.6 and with the change of the order of summation we have

$$(2.6) \quad \begin{aligned} \left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r &\geq C \sum_{\nu=1}^{\infty} A(\nu) \sum_{n=1}^{\nu} n^{-mr} |a_n|^r \\ &= C \sum_{n=1}^{\infty} n^{-mr} |a_n|^r \sum_{\nu=n}^{\infty} A(\nu) \\ &\geq C \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n). \end{aligned}$$

Relations (2.5) and (2.6) prove Theorem 2.2. □

We can deduce three corollaries from Theorem 2.1 and Theorem 2.2.

Corollary 2.3. Under the conditions of Theorem 2.1 and with $b(n) \leq CA(n)$, we have

$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r(1-\frac{2}{p})} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}}.$$



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Corollary 2.4. Under the conditions of Theorem 2.2 and with $b_1(n) \leq CA(n)$, we have

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \right\}^{\frac{1}{r}}.$$

As a special case, for $\alpha(t) = t^{-\beta r - 1}$, it is easy to prove the estimates:

$$A(n) \leq Cn^{\beta r - 1} \quad \text{and} \quad b(n) \leq Cn^{\beta r}.$$

From Theorem 2.1 and the last estimates we can deduce the following result proved in [1].

Corollary 2.5 ([1]). Let m be any natural number and

$$0 < \beta \leq m, \quad 1 < p < +\infty, \quad 1 \leq r < +\infty, \quad 1/p + 1/q = 1.$$

If the coefficients of series (1.1) or (1.2) satisfy $\sum_{k=1}^{\infty} |\Delta a_k| < +\infty$, then

$$\|f\|_{\beta,p,r}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r(\beta + \frac{1}{q}) - 1} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \right\}^{\frac{1}{r}}.$$

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