ON A GENERALIZATION OF LIPSCHITZ'S CLASSES

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Received: 26 March, 2008

Accepted: 06 August, 2008

Communicated by: H. Bor

2000 AMS Sub. Class.: 42A20, 42A32.

Key words: Lipschitz classes, Fourier series.

Abstract: In this paper we obtain a generalization of Lipschitz's classes $\Lambda^m(\beta, p, r)$ defined in [1]. We give necessary conditions for even or odd functions with Fourier series to belong to the classes $\Lambda^m(p, r, \alpha)$. We also give sufficient conditions for even or odd functions with Fourier series to belong to the same classes.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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vol. 9, iss. 3, art. 73, 2008

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Definitions and Useful Statements

We consider the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

or

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

where a_n are Fourier coefficients of integrable function f.

Definition 1.1. *We say that a function f belongs to WA^p, $(1 < p < \infty)$ if*

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p < +\infty$$

where $\Delta a_k = a_k - a_{k+1}$ (see [1]).

We say that any function $\alpha(t)$ is a function of type σ (see [4]) if it is measurable in $[0,1]$, integrable in $[\delta, 1]$ for each $\delta \in (0,1)$, and there exist real numbers $C_{1,\alpha} > 0$, σ and $\delta_0 \in (0,1)$ such that

1. $\alpha(t) \geq C_{1,\alpha}$, for all $t \in [0,1]$;
2. $\int_0^\delta \alpha(t) t^s dt < \infty$ for each $s > \sigma$ and $\delta \in (0, \delta_0)$;
3. $\int_0^\delta \alpha(t) t^s dt = \infty$ for each $s < \sigma$ and $\delta \in (0, \delta_0)$, and

$$\int_0^\delta \alpha(t) t^\sigma dt \leq C_2 \delta^\sigma \int_\delta^{2\delta} \alpha(t) dt.$$

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issn: 1443-5756

In [1], the classes $\Lambda^m(\beta, p, r,)$ are defined in the following way

Definition 1.2. $f \in \Lambda^m(\beta, p, r,)$, *if*

$$\|f\|_{\beta,p,r}^{(m)} \equiv \left\{ \int_0^1 \left[\int_0^{2\pi} \frac{|\Delta_m f(x,t)|^p}{t^{\beta p}} dx \right]^{\frac{r}{p}} \frac{dt}{t} \right\}^{\frac{1}{r}} < +\infty,$$

where $1 < p < +\infty$, $1 \le r < +\infty$, $\beta > 0$, $m \in \mathbb{N}$ *and*

$$\Delta_m f(x,t) = \sum_{i=1}^{m} (-1)^i C_m^i f[x + (m - 2i)t].$$

Now we define classes $\Lambda^m(p, r, \alpha)$ as follows:

Definition 1.3. *We say* $f \in \Lambda^m(p, r, \alpha)$, *if*

$$\|f\|_{p,r,\alpha}^{(m)} \equiv \left\{ \int_0^1 \alpha(t) \left[\int_0^{2\pi} |\Delta_m f(x,t)|^p \, dx \right]^{\frac{r}{p}} dt \right\}^{\frac{1}{r}} < +\infty,$$

where $\alpha(t)$ *is function of the type* σ.

For $\alpha(t) = t^{-r\beta-1}, \beta > 0$, we get the classes $\Lambda^m(\beta, p, r)$, considered in [1]. Therefore classes $\Lambda^m(p, r, \alpha)$ are generalizations of classes $\Lambda^m(\beta, p, r)$.

We need some auxiliary statements.

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issn: 1443-5756

Lemma 1.4 ([2]). *Let* a_ν*,* b_ν *and* β_n *be numbers such that* $a_\nu \geq 0$*,* $b_\nu \geq 0$ *and* $\sum_{\nu=n}^{\infty} a_\nu = a_n\beta_n$*:*

1. *For* $0 < p \leq 1$ *the following inequality is valid*

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=1}^{\nu} b_\mu\right)^p \geq p^p \sum_{\nu=1}^{\infty} a_\nu \left(b_\nu\beta_\nu\right)^p;$$

2. *For* $1 \leq p < \infty$ *we have*

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=1}^{\nu} b_\mu\right)^p \leq p^p \sum_{\nu=1}^{\infty} a_\nu \left(b_\nu\beta_\nu\right)^p.$$

Lemma 1.5 ([2]). *Let* a_ν*,* b_ν *and* γ_n *be numbers such that* $a_\nu \geq 0$*,* $b_\nu \geq 0$ *and* $\sum_{\nu=1}^{n} a_\nu = b_n\gamma_n$*:*

1. *For* $0 < p \leq 1$*, we have*

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=\nu}^{\infty} b_\mu\right)^p \geq p^p \sum_{\nu=1}^{\infty} a_\nu \left(b_\nu\gamma_\nu\right)^p;$$

2. *For* $1 \leq p < \infty$*, we have*

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=\nu}^{\infty} b_\mu\right)^p \leq p^p \sum_{\nu=1}^{\infty} a_\nu \left(b_\nu\gamma_\nu\right)^p.$$

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issn: 1443-5756

Lemma 1.6 ([3]). *Let* μ, τ *and* a_ν *be numbers such that* $0 < \mu < \tau < \infty$ *and* $a_\nu \geq 0$. *Then*

$$\left(\sum_{\nu=1}^{\infty} a_\nu^\tau\right)^{\frac{1}{\tau}} \leq \left(\sum_{\nu=1}^{\infty} a_\nu^\mu\right)^{\frac{1}{\mu}}.$$

We denote by C a constant that depends only on m, p, r and may be different in different relations.

Theorem 1.7 ([1]). *If* $f \in WA^p$, $1 < p < +\infty$, *then*

$$\left\{\omega_p^{(m)}(h; f)\right\}^p \leq Ch^{mp} \sum_{n \leq \left[\frac{1}{h}\right]} n^{(m+1)p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^p + C \sum_{n > \left[\frac{1}{h}\right]} n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^p,$$

where $\omega_p^{(m)}(h; f)$ *is the integral modulus of smoothness of order* m.

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2. Main Results

Let us denote

$$A(n) := \int_{1/(n+1)}^{1/n} \alpha(t)dt,$$

$$b(n) := b_1(n) + b_2(n) = n^{mr} \int_0^{1/n} \alpha(t)t^{mr}dt + \int_{1/(n+1)}^{1} \alpha(t)dt.$$

We have the following first main result.

Theorem 2.1. *Let m be any natural number and*

$$f \in AW^p, \quad 1 < p < +\infty, \quad 1 \le r < +\infty.$$

If for the coefficients of series (1.1) or (1.2) we have $\sum_{k=1}^{\infty} |\Delta a_k| < +\infty$, then:

1. *For $p \le r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \le C \left\{ \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \left(\frac{b(n)}{A(n)} \right)^{\frac{r}{p}-1} \right\}^{\frac{1}{r}};$$

2. *For $p > r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \le C \left\{ \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}}.$$

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issn: 1443-5756

Proof. Using the characteristics of the integral modulus of smoothness we have

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r = \int_0^1 \alpha(t)\left[\int_0^{2\pi} |\Delta_m f(x,t)|^p\,dx\right]^{\frac{r}{p}} dt$$
$$\leq \sum_{N=1}^{\infty} \int_{1/(N+1)}^{1/N} \alpha(t)\left[\omega_p^{(m)}(f;t)\right]^r dt$$
$$\leq \sum_{N=1}^{\infty} \left[\omega_p^{(m)}(f;1/N)\right]^r \int_{1/(N+1)}^{1/N} \alpha(t)dt$$
$$= \sum_{N=1}^{\infty} A(N)\left[\omega_p^{(m)}(f;1/N)\right]^r.$$

According to the Theorem 1.7, we have

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \leq C\sum_{N=1}^{\infty} A(N)N^{-mr}\left\{\sum_{n=1}^{N} n^{(m+1)p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right\}^{\frac{r}{p}}$$
$$+ C\sum_{N=1}^{\infty} A(N)\left\{\sum_{n=N+1}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right\}^{\frac{r}{p}}$$
$$= I_1 + I_2.$$

Now we estimate I_1 and I_2. Let $r/p \geq 1$. Then according to Lemma 1.4 we have

$$I_1 \leq C\sum_{N=1}^{\infty} A(N)N^{-mr}\left\{N^{(m+1)p-2}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^p \beta_N\right\}^{\frac{r}{p}}.$$

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issn: 1443-5756

Now we estimate the quantity β_N:

$$\begin{aligned} A(N)N^{-mr}\beta_N &= \sum_{i=N}^{\infty} A(i)i^{-mr} \\ &= \sum_{i=N}^{\infty} \left(\frac{i+1}{i}\right)^{mr} \cdot \frac{1}{(i+1)^{mr}} \int_{1/(i+1)}^{1/i} \alpha(t)dt \\ &\leq 2^{mr} \int_0^{1/N} \alpha(t)t^{mr}dt, \end{aligned}$$

or

$$\beta_N \leq C\frac{b_1(N)}{A(N)}.$$

Consequently

$$I_1 \leq C\sum_{N=1}^{\infty} A(N)N^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^r \left\{\frac{b_1(N)}{A(N)}\right\}^{\frac{r}{p}}. \tag{2.1}$$

According to Lemma 1.5, for $r/p \geq 1$ we have

$$I_2 \leq C\sum_{N=1}^{\infty} A(N)\left\{N^{p-2}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^p \gamma_N\right\}^{\frac{r}{p}}.$$

We estimate the quantity γ_N:

$$A(N)\gamma_N = \sum_{i=1}^{N} A(i) = \int_{1/(N+1)}^{1} \alpha(t)dt \Rightarrow \gamma_N = \frac{b_2(N)}{A(N)}.$$

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Consequently

$$I_2 \le C \sum_{N=1}^{\infty} A(N) N^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=N}^{\infty} |\Delta a_k|\right)^r \left\{\frac{b_2(N)}{A(N)}\right\}^{\frac{r}{p}}. \tag{2.2}$$

By (2.1) and (2.2) we get

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \le C \sum_{N=1}^{\infty} A(N) N^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=N}^{\infty} |\Delta a_k|\right)^r \left\{\left(\frac{b_1(N)}{A(N)}\right)^{\frac{r}{p}} + \left(\frac{b_2(N)}{A(N)}\right)^{\frac{r}{p}}\right\}.$$

Finally, according to Lemma 1.6, for $r \ge p$ we have

$$\|f\|_{p,r,\alpha}^{(m)} \le C \left\{\sum_{N=1}^{\infty} A(N) N^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=N}^{\infty} |\Delta a_k|\right)^r \left(\frac{b(N)}{A(N)}\right)^{\frac{r}{p}}\right\}^{\frac{1}{r}}.$$

Now let $r/p < 1$. Then, according to Lemma 1.6, we have

$$I_1 \le C \sum_{N=1}^{\infty} A(N) N^{-mr} \sum_{n=1}^{N} n^{(m+1)r-\frac{2r}{p}} \left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^r.$$

If we change the order of summation we get

$$\begin{aligned} I_1 &\le C \sum_{n=1}^{\infty} n^{(m+1)r-\frac{2r}{p}} \left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^r \sum_{N=n}^{\infty} A(N) N^{-mr} \\ &\le C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^r b_1(n). \end{aligned} \tag{2.3}$$

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issn: 1443-5756

Now we estimate I_2. Using Lemma 1.6 and changing the order of summation we have:

$$I_2 \leq C \sum_{N=1}^{\infty} A(N) \sum_{n=N}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r$$
$$= C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \sum_{N=1}^{n} A(N)$$
$$(2.4) \qquad = C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b_2(n).$$

From (2.3) and (2.4) we deduce

$$\|f\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}},$$

which fully demonstrates Theorem 2.1.

□

Theorem 2.2. *Let m be any natural number and*

$$1 < p \leq 2, \quad 1 \leq r < +\infty, \quad 1/p + 1/q = 1.$$

If a_n are the coefficients of series (1.1) or (1.2), then:

1. *For $r \leq q$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \left[\frac{b_1(n)}{A(n)} \right]^{\frac{r}{q}-1} \right\}^{\frac{1}{r}};$$

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issn: 1443-5756

2. For $r > q$ we have

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \right\}^{\frac{1}{r}}.$$

Proof. Let f be an even function. If f is an odd function then the proof of the theorem is analogous to the even case. It is not difficult to see that the Fourier series of $\Delta_m f(x,t)$ is

$$\Delta_m f(x,t) \sim \begin{cases} (-1)^{\frac{m}{2}} 2^m \sum_{n=1}^{\infty} a_n \cos nx \sin^m nt, & \textit{for } m \textit{ even} \\ (-1)^{\frac{m-1}{2}-1} 2^m \sum_{n=1}^{\infty} a_n \sin nx \sin^m nt, & \textit{for } m \textit{ odd}. \end{cases}$$

According to the well-known Hausdorf-Young's theorem we find

$$C \left(\int_0^{2\pi} |\Delta_m f(x,t)|^p \, dx \right)^{\frac{r}{p}} \geq \left(\sum_{n=1}^{\infty} |a_n|^q |\sin nt|^{mq} \right)^{\frac{r}{q}},$$

and then

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} \int_{1/(\nu+1)}^{1/\nu} \alpha(t) \left(\sum_{n=1}^{\nu} |a_n|^q |\sin nt|^{mq} \right)^{\frac{r}{q}} dt.$$

Using the well-known inequality $\sin B \geq \frac{2}{\pi} B$ for $0 \leq B \leq \frac{\pi}{2}$, we get

$$\left\{ \|f\|_{p,r,\alpha}^{(m)} \right\}^r \geq C \sum_{\nu=1}^{\infty} A(\nu) \left(\sum_{n=1}^{\nu} n^{-mq} |a_n|^q \right)^{\frac{r}{q}}.$$

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issn: 1443-5756

Let $r \le q$, then according to Lemma 1.4 we have

$$\left\{\left\|f\right\|_{p,r,\alpha}^{(m)}\right\}^r \ge C\sum_{\nu=1}^{\infty} A(\nu)\left[\nu^{-mq}\left|a_\nu\right|^q \beta_\nu\right]^{\frac{r}{q}}.$$

It is easy to prove that $\beta_\nu \ge \frac{b_1(\nu)}{A(\nu)}$, from which we get

$$\left\{\left\|f\right\|_{p,r,\alpha}^{(m)}\right\}^r \ge C\sum_{\nu=1}^{\infty} \nu^{-mr}\left|a_\nu\right|^r b_1(\nu)\left[\frac{b_1(\nu)}{A(\nu)}\right]^{\frac{r}{q}-1}. \tag{2.5}$$

Let $q < r$, then according to Lemma 1.6 and with the change of the order of summation we have

$$\begin{aligned}\left\{\left\|f\right\|_{p,r,\alpha}^{(m)}\right\}^r &\ge C\sum_{\nu=1}^{\infty} A(\nu)\sum_{n=1}^{\nu} n^{-mr}\left|a_n\right|^r \\ &= C\sum_{n=1}^{\infty} n^{-mr}\left|a_n\right|^r \sum_{\nu=n}^{\infty} A(\nu) \\ &\ge C\sum_{n=1}^{\infty} n^{-mr}\left|a_n\right|^r b_1(n). \qquad (2.6)\end{aligned}$$

Relations (2.5) and (2.6) prove Theorem 2.2. □

We can deduce three corollaries from *Theorem 2.1* and *Theorem 2.2*.

Corollary 2.3. *Under the conditions of Theorem 2.1 and with $b(n) \le CA(n)$, we have*

$$\left\|f\right\|_{p,r,\alpha}^{(m)} \le C\left\{\sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=n}^{\infty}\left|\Delta a_k\right|\right)^r b(n)\right\}^{\frac{1}{r}}.$$

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issn: 1443-5756

Corollary 2.4. *Under the conditions of Theorem 2.2 and with $b_1(n) \leq CA(n)$, we have*

$$\|f\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n) \right\}^{\frac{1}{r}}.$$

As a special case, for $\alpha(t) = t^{-\beta r-1}$, it is easy to prove the estimates:

$$A(n) \leq Cn^{\beta r-1} \quad \text{and} \qquad b(n) \leq Cn^{\beta r}.$$

From Theorem 2.1 and the last estimates we can deduce the following result proved in [1].

Corollary 2.5 ([1]). *Let m be any natural number and*

$$0 < \beta \leq m, \quad 1 < p < +\infty, \quad 1 \leq r < +\infty, \quad 1/p + 1/q = 1.$$

If the coefficients of series (1.1) or (1.2) satisfy $\sum_{k=1}^{\infty} |\Delta a_k| < +\infty$, then

$$\|f\|_{\beta,p,r}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r\left(\beta+\frac{1}{q}\right)-1} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^r \right\}^{\frac{1}{r}}.$$

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issn: 1443-5756