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A NOTE ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS

L. LEINDLER

Bolyai Institute, University of Szeged
Aradi vértanúk tere 1
H-6720 Szeged, Hungary.

EMail: leindler@math.u-szeged.hu

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Abstract

A crucial assumption of a previous theorem of the author is omitted without changing the consequence. This is achieved by proving a new (?) estimation on the absolute value of the terms of a real sequence by means of the sums of the differences of the terms.

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Contents

1. Introduction

In [4] we proved a theorem on absolute Riesz summability. Our paper was initiated by a theorem of H. Bor [2] (see also [3]). Now we do not intend to recall these theorems, the interested readers are referred to [4]. The aim of the present note is to show that the crucial condition of our proof, $\lambda_n \to 0$, can be deduced from two other conditions of the theorem.

In order to provide the new theorem we require some notions and notations.

A positive sequence $\{a_n\}$ is said to be *quasi increasing* if there exists a constant $K = K(\{a_k\}) \geq 1$ such that

$$K\, a_n \geq a_m \tag{1.1}$$

holds for all $n \geq m$.

The series $\sum_{n=1}^{\infty} a_n$ with partial sums s_n is said to be summable $|\overline{N}, p_n|_k,\ k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where $\{p_n\}$ is a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty,$$

and

$$t_n := \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu\, s_\nu.$$

2. Result

As we have written above, the new theorem to be presented here deviates from our previous result merely that an assumption, $\lambda_n \to 0$, does not appear among the conditions.

The new theorem reads as follows.

Theorem 2.1. *Let $\{\lambda_n\}$ be a sequence of real numbers satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| < \infty. \tag{2.1}$$

Suppose that there exists a positive quasi increasing sequence $\{X_n\}$ such that

$$\sum_{n=1}^{\infty} X_n |\Delta\, \lambda_n| < \infty, \qquad (\Delta\, \lambda_n := \lambda_n - \lambda_{n+1}), \tag{2.2}$$

$$X_m^* := \sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m), \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \tag{2.4}$$

and

$$\sum_{n=1}^{\infty} n\, X_n^* |\Delta(|\Delta\, \lambda_n|)| < \infty \tag{2.5}$$

hold. Then the series $\sum a_n\, \lambda_n$ is summable $|\overline{N}, p_n|_k,\ k \geq 1$.

It is clear that if we can verify first that the conditions (2.1) and (2.2) imply that $\lambda_n \to 0$, then the proof given in [4] is acceptable now, too. We shall follow this way.

3. Lemmas

We need the following lemmas for the proof our statement.

Lemma 3.1 ([1, 2.2.2. p. 72]). *If $\{\mu_n\}$ is a positive, monotone increasing and tending to infinity sequence, then the convergence of the series $\sum u_n \mu_n^{-1}$ implies the estimate*

$$\sum_{k=1}^{n} u_k = o(\mu_n). \tag{3.1}$$

This lemma is the famous Kronecker lemma.

Lemma 3.2. *Let $\{\gamma_n\}$ be a sequence of real numbers and denote*

$$\Gamma_n := \sum_{k=1}^{n} \gamma_k \quad \textit{and} \quad R_n := \sum_{k=n}^{\infty} |\Delta \gamma_k|.$$

If $\Gamma_n = o(n)$ then there exists a natural number n_0 such that

$$|\gamma_n| \le 2R_n \tag{3.2}$$

for all $n \ge n_0$. Naturally $R_1 < \infty$ is assumed, otherwise (3.2) is a triviality. However then $\Gamma_n = o(n)$ is not only sufficient but also necessary to (3.2).

Remark 1. *It is clear that if $\gamma_n \to 0$ then $|\gamma_n| \le R_n$ is trivial, but not if $\gamma_n \not\to 0$, see e.g. $\gamma_n = c \ne 0$ or $\gamma_n = 2 - \frac{1}{n}$. Perhaps (3.2) is known, but unfortunately I have not encountered it in any paper. I presume that (3.2) is not very known, namely recently two papers used it without the assumption $\Gamma_n = o(n)$, or its consequences to be given next.*

4. Corollaries

Lemma 3.2 implies the following usable consequences.

Corollary 4.1. *Let $\{\rho_n\}$ be a sequence of real numbers. If $\rho_n = o(n)$ then*

$$|\Delta\,\rho_n| \le 2\sum_{k=n}^{\infty} |\Delta^2\,\rho_k|, \quad (\Delta^2\,\rho_k = \Delta(\Delta\,\rho_k)). \tag{4.1}$$

holds if n is large enough.

Corollary 4.2. *Let $\alpha \ge 0$ and $\{\rho_n\}$ be as in Corollary 4.1. If*

$$\sum_{k=1}^{\infty} k^{\alpha}|\Delta^2\,\rho_k| < \infty, \quad (\rho_n = o(n)), \tag{4.2}$$

then

$$|\Delta\,\rho_n| = o(n^{-\alpha}). \tag{4.3}$$

In my view Lemma 3.2 and these corollaries are of independent interest.

5. Proofs

Proof of Lemma 3.2. Let us assume that (3.2) does not hold for any n_0. Then there exists an increasing sequence $\{\nu_n\}$ of the natural numbers such that

$$2R_{\nu_n} < |\gamma_{\nu_n}|. \tag{5.1}$$

Let $m = \nu_n$, and be fixed. Then for any $k > m$

$$2R_m < |\gamma_m| = \left|\sum_{i=m}^{k-1} \Delta\,\gamma_i + \gamma_k\right| \leq R_m + |\gamma_k|,$$

whence

$$R_m < |\gamma_k| \tag{5.2}$$

holds.

Now let us choose n such that

$$(n-m)R_m > 2|\Gamma_m|. \tag{5.3}$$

It is easy to verify that for all $k > m$ the terms γ_k have the same sign, that is, $\gamma_k \cdot \gamma_{k+1} > 0$. Namely if γ_k and γ_{k+1} have different sign then, by (5.2), $|\Delta\,\gamma_k| > 2R_m$. But this contradicts the fact that $R_m \geq R_k \geq |\Delta\,\gamma_k|$.

Thus, if $n > m$ then

$$\Gamma_n = \Gamma_m + \sum_{k=m+1}^{n} \gamma_k,$$

and by invoking inequalities (5.2) and (5.3) we obtain that

$$|\Gamma_n| \geq \sum_{k=m+1}^{n} |\gamma_k| - |\Gamma_m| \geq \frac{1}{2}(n-m)R_m.$$

Since the last inequality opposes the assumption $\Gamma_n = o(n)$, thus (3.2) is proved. To verify the necessity of the condition $\Gamma_n = o(n)$ it suffices to observe that $R_1 < \infty$ implies $R_n \to 0$, thus, by (3.2),

$$\frac{1}{n}\sum_{k=1}^{n} |\gamma_k| \to 0$$

clearly holds. □

Proof of Corollary 4.1. Applying Lemma 3.2 with $\gamma_n := \Delta\,\rho_n$, we promptly get the statement of Corollary 4.1. □

Proof of Corollary 4.2. In view of (4.2) it is plain that

$$\sum_{k=n}^{\infty} |\Delta^2\,\rho_k| = o(n^{-\alpha}),$$

whence (4.3) follows by (4.1). □

Proof of Theorem 2.1. It is clearly sufficient to verify that the conditions (2.1) and (2.2) imply that

$$\lambda_n \to 0, \tag{5.4}$$

namely with this additional condition the assertion of Theorem 2.1 had been proved in [4].

Now we prove (5.4). In view of Lemma 3.1 we know that $\sum_{k=1}^{n}|\lambda_k| = o(n)$, thus the assumptions of Lemma 3.2 are satisfied with $\gamma_n := \lambda_n$. Furthermore the condition (2.2) visibly implies that

$$\sum_{k=n}^{\infty} |\Delta\,\lambda_k| = o(1), \tag{5.5}$$

thus (3.2), by (5.5), proves (5.4).

The proof is complete. $\square$

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