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Abstract

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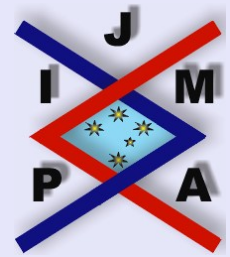


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Abstract

A crucial assumption of a previous theorem of the author is omitted without changing the consequence. This is achieved by proving a new (?) estimation on the absolute value of the terms of a real sequence by means of the sums of the differences of the terms.

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1. Introduction

In [4] we proved a theorem on absolute Riesz summability. Our paper was initiated by a theorem of H. Bor [2] (see also [3]). Now we do not intend to recall these theorems, the interested readers are referred to [4]. The aim of the present note is to show that the crucial condition of our proof, $\lambda_n \rightarrow 0$, can be deduced from two other conditions of the theorem.

In order to provide the new theorem we require some notions and notations.

A positive sequence $\{a_n\}$ is said to be *quasi increasing* if there exists a constant $K = K(\{a_k\}) \geq 1$ such that

$$(1.1) \quad K a_n \geq a_m$$

holds for all $n \geq m$.

The series $\sum_{n=1}^{\infty} a_n$ with partial sums s_n is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where $\{p_n\}$ is a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^n p_{\nu} \rightarrow \infty,$$

and

$$t_n := \frac{1}{P_n} \sum_{\nu=0}^n p_{\nu} s_{\nu}.$$



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2. Result

As we have written above, the new theorem to be presented here deviates from our previous result merely that an assumption, $\lambda_n \rightarrow 0$, does not appear among the conditions.

The new theorem reads as follows.

Theorem 2.1. *Let $\{\lambda_n\}$ be a sequence of real numbers satisfying the condition*

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| < \infty.$$

Suppose that there exists a positive quasi increasing sequence $\{X_n\}$ such that

$$(2.2) \quad \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \quad (\Delta \lambda_n := \lambda_n - \lambda_{n+1}),$$

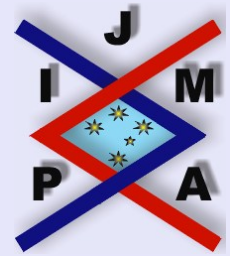
$$(2.3) \quad X_m^* := \sum_{n=1}^m \frac{1}{n} |t_n|^k = O(X_m),$$

$$(2.4) \quad \sum_{n=1}^m \frac{p_n}{P_n} |t_n|^k = O(X_m)$$

and

$$(2.5) \quad \sum_{n=1}^{\infty} n X_n^* |\Delta(|\Delta \lambda_n|)| < \infty$$

hold. Then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k, k \geq 1$.



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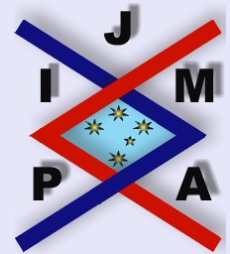
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It is clear that if we can verify first that the conditions (2.1) and (2.2) imply that $\lambda_n \rightarrow 0$, then the proof given in [4] is acceptable now, too. We shall follow this way.



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3. Lemmas

We need the following lemmas for the proof our statement.

Lemma 3.1 ([1, 2.2.2. p. 72]). *If $\{\mu_n\}$ is a positive, monotone increasing and tending to infinity sequence, then the convergence of the series $\sum u_n \mu_n^{-1}$ implies the estimate*

$$(3.1) \quad \sum_{k=1}^n u_k = o(\mu_n).$$

This lemma is the famous Kronecker lemma.

Lemma 3.2. *Let $\{\gamma_n\}$ be a sequence of real numbers and denote*

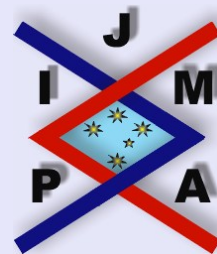
$$\Gamma_n := \sum_{k=1}^n \gamma_k \quad \text{and} \quad R_n := \sum_{k=n}^{\infty} |\Delta \gamma_k|.$$

If $\Gamma_n = o(n)$ then there exists a natural number n_0 such that

$$(3.2) \quad |\gamma_n| \leq 2R_n$$

for all $n \geq n_0$. Naturally $R_1 < \infty$ is assumed, otherwise (3.2) is a triviality. However then $\Gamma_n = o(n)$ is not only sufficient but also necessary to (3.2).

Remark 1. *It is clear that if $\gamma_n \rightarrow 0$ then $|\gamma_n| \leq R_n$ is trivial, but not if $\gamma_n \not\rightarrow 0$, see e.g. $\gamma_n = c \neq 0$ or $\gamma_n = 2 - \frac{1}{n}$. Perhaps (3.2) is known, but unfortunately I have not encountered it in any paper. I presume that (3.2) is not very known, namely recently two papers used it without the assumption $\Gamma_n = o(n)$, or its consequences to be given next.*



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4. Corollaries

Lemma 3.2 implies the following usable consequences.

Corollary 4.1. *Let $\{\rho_n\}$ be a sequence of real numbers. If $\rho_n = o(n)$ then*

$$(4.1) \quad |\Delta \rho_n| \leq 2 \sum_{k=n}^{\infty} |\Delta^2 \rho_k|, \quad (\Delta^2 \rho_k = \Delta(\Delta \rho_k)).$$

holds if n is large enough.

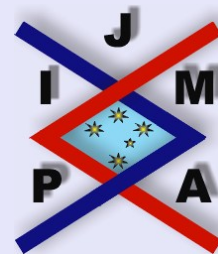
Corollary 4.2. *Let $\alpha \geq 0$ and $\{\rho_n\}$ be as in Corollary 4.1. If*

$$(4.2) \quad \sum_{k=1}^{\infty} k^\alpha |\Delta^2 \rho_k| < \infty, \quad (\rho_n = o(n)),$$

then

$$(4.3) \quad |\Delta \rho_n| = o(n^{-\alpha}).$$

In my view Lemma 3.2 and these corollaries are of independent interest.



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5. Proofs

Proof of Lemma 3.2. Let us assume that (3.2) does not hold for any n_0 . Then there exists an increasing sequence $\{\nu_n\}$ of the natural numbers such that

$$(5.1) \quad 2R_{\nu_n} < |\gamma_{\nu_n}|.$$

Let $m = \nu_n$, and be fixed. Then for any $k > m$

$$2R_m < |\gamma_m| = \left| \sum_{i=m}^{k-1} \Delta \gamma_i + \gamma_k \right| \leq R_m + |\gamma_k|,$$

whence

$$(5.2) \quad R_m < |\gamma_k|$$

holds.

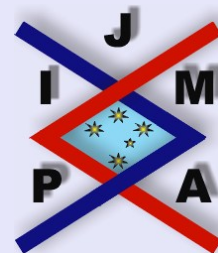
Now let us choose n such that

$$(5.3) \quad (n - m)R_m > 2|\Gamma_m|.$$

It is easy to verify that for all $k > m$ the terms γ_k have the same sign, that is, $\gamma_k \cdot \gamma_{k+1} > 0$. Namely if γ_k and γ_{k+1} have different sign then, by (5.2), $|\Delta \gamma_k| > 2R_m$. But this contradicts the fact that $R_m \geq R_k \geq |\Delta \gamma_k|$.

Thus, if $n > m$ then

$$\Gamma_n = \Gamma_m + \sum_{k=m+1}^n \gamma_k,$$



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and by invoking inequalities (5.2) and (5.3) we obtain that

$$|\Gamma_n| \geq \sum_{k=m+1}^n |\gamma_k| - |\Gamma_m| \geq \frac{1}{2}(n-m)R_m.$$

Since the last inequality opposes the assumption $\Gamma_n = o(n)$, thus (3.2) is proved. To verify the necessity of the condition $\Gamma_n = o(n)$ it suffices to observe that $R_1 < \infty$ implies $R_n \rightarrow 0$, thus, by (3.2),

$$\frac{1}{n} \sum_{k=1}^n |\gamma_k| \rightarrow 0$$

clearly holds. □

Proof of Corollary 4.1. Applying Lemma 3.2 with $\gamma_n := \Delta \rho_n$, we promptly get the statement of Corollary 4.1. □

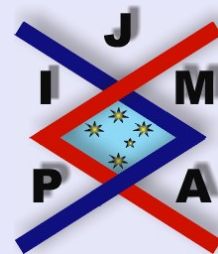
Proof of Corollary 4.2. In view of (4.2) it is plain that

$$\sum_{k=n}^{\infty} |\Delta^2 \rho_k| = o(n^{-\alpha}),$$

whence (4.3) follows by (4.1). □

Proof of Theorem 2.1. It is clearly sufficient to verify that the conditions (2.1) and (2.2) imply that

$$(5.4) \quad \lambda_n \rightarrow 0,$$



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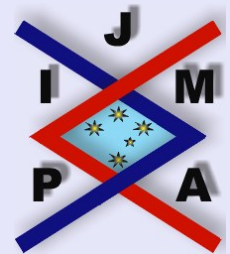
namely with this additional condition the assertion of Theorem 2.1 had been proved in [4].

Now we prove (5.4). In view of Lemma 3.1 we know that $\sum_{k=1}^n |\lambda_k| = o(n)$, thus the assumptions of Lemma 3.2 are satisfied with $\gamma_n := \lambda_n$. Furthermore the condition (2.2) visibly implies that

$$(5.5) \quad \sum_{k=n}^{\infty} |\Delta \lambda_k| = o(1),$$

thus (3.2), by (5.5), proves (5.4).

The proof is complete. □



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