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# INEQUALITIES DEFINING CERTAIN SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS INVOLVING FRACTIONAL CALCULUS OPERATORS

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## Abstract

Making use of a certain fractional calculus operator, we introduce two new subclasses $M_\delta(p;\lambda,\mu,\eta)$ and $T_\delta(p;\lambda,\mu,\eta)$ of analytic and $p-$valent functions in the open unit disk. The results investigated exhibit the sufficiency conditions for a function to belong to each of these classes. Several geometric properties of such multivalent functions follow, and these consequences are also mentioned.

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# Contents

# 1. Introduction and Definitions

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$.

A function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently starlike in $\mathcal{U}$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathcal{U}), \tag{1.2}$$

and the function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently convex in $\mathcal{U}$, if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.3}$$

Further, a function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently close-to-convex in $\mathcal{U}$, if

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.4}$$

One may refer to [1], [2] and [9] for above definitions and other related details.

The operator $J_{0,z}^{\lambda,\mu,\eta}$ occurring in this paper is the Saigo type fractional calculus operator defined as follows ([8]):

**Definition 1.1.** *Let* $0 \leq \lambda < 1$ *and* $\mu,\eta \in \mathbb{R}$, *then*

$$(1.5)\quad J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-t)^{-\lambda} F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z}\right) f(t)dt\right),$$

*where the function* $f(z)$ *is analytic in a simply-connected region of the* $z$*-plane containing the origin, with the order*

$$f(z) = O\left(|z|^{\varepsilon}\right)(z \to 0), \quad \textit{where } \varepsilon > \max\{0, \mu-\eta\} - 1.$$

It being understood that $(z-t)^{-\lambda}$ denotes the principal value for $0 \leq \arg(z-t) < 2\pi$. The ${}_2F_1$ function occurring in the right-hand side of (1.5) is the familiar Gaussian hypergeometric function (see [9] for its definition).

**Definition 1.2.** *Under the hypotheses of Definition 1.1, a fractional calculus operator* $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ *is defined by ([7])*

$$(1.6)\quad J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (z \in \mathcal{U}; m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}).$$

We observe that

$$(1.7)\quad D_z^{\lambda} f(z) = J_{0,z}^{\lambda,\lambda,\eta} f(z) \qquad (0 \leq \lambda < 1),$$

and

$$(1.8)\quad D_z^{\lambda+m} f(z) = J_{0,z}^{\lambda+m,\lambda+m,\eta+m} f(z) \qquad (0 \leq \lambda < 1; m \in \mathbb{N}_0),$$

where $D_z^{\lambda+m}$ is the well known fractional derivative operator ([6], [9]).

We introduce here two subclasses of functions $\mathcal{M}_\delta(p;\lambda,\mu,\eta)$ and $\mathcal{T}_\delta(p;\lambda,\mu,\eta)$ which are defined as follows.

**Definition 1.3.** *Let* $\delta \in \mathbb{R}\backslash\{0\}, p \in \mathbb{N}, 0 \leq \lambda < 1, \mu < 1,$ *and* $\eta > \max(\lambda, \mu) - p - 1$. *Then the function* $f(z) \in \mathcal{A}_p$ *is said to belong to* $\mathcal{M}_\delta(p; \lambda, \mu, \eta)$ *if it satisfies the inequality*

$$\left| \left( \frac{z J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \right)^\delta - (p-\mu)^\delta \right| < (p-\mu)^\delta \qquad (z \in \mathcal{U}), \tag{1.9}$$

*where the value of* $\left( z\, J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) / J_{0,z}^{\lambda,\mu,\eta} f(z) \right)^\delta$ *is taken as its principal value.*

**Definition 1.4.** *Under the hypotheses of Definition 1.3, the function* $f(z) \in \mathcal{A}_p$ *is said to belong to* $\mathcal{T}_\delta(p; \lambda, \mu, \eta)$ *if it satisfies the inequality*

$$\left| \left( z^{\mu-p} J_{0,z}^{\lambda,\mu,\eta} f(z) \right)^\delta - \left( \frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)} \right)^\delta \right| < \left( \frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)} \right)^\delta \qquad (z \in \mathcal{U}), \tag{1.10}$$

*where the value of* $\left( z^{\mu-p} J_{0,z}^{\lambda,\mu,\eta} f(z) \right)^\delta$ *is considered to be its principal value.*

*For* $\lambda = \mu$, *we have*

$$\mathcal{M}_\delta(p; \mu, \mu, \eta) = \mathcal{M}_\delta(p; \mu), \tag{1.11}$$

*and*

$$\mathcal{T}_\delta(p; \mu, \mu, \eta) = \mathcal{T}_\delta(p; \mu). \tag{1.12}$$

The classes $\mathcal{M}_\delta(p;\mu)$and $\mathcal{T}_\delta(p;\mu)$ were studied recently in [4]. In view of the operational relation (1.8), it may be noted that the functions in $\mathcal{M}_1(p;0)$ are $p-$valently starlike in $\mathcal{U}$, whereas, the functions in$\mathcal{T}_1(p;1)$ are $p-$valently close-to-convex in $\mathcal{U}$.

In this paper we investigate characterization properties giving sufficiency conditions for functions of the form (1.1) to belong to the classes $\mathcal{M}_\delta(p;\lambda,\mu,\eta)$ and $\mathcal{T}_\delta(p;\lambda,\mu,\eta)$ involving the fractional calculus operator (1.6). Several consequences of the main results and their relevance to known results are also pointed out.

## 2. Results Required

We mention the following results which are used in the sequel:

**Lemma 2.1.** *([8]). If* $0 \leq \lambda < 1$*;* $\mu,\eta \in \mathbb{R}$ *and* $k > \max\{0, \mu - \eta\} - 1$*, then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(1+k)\Gamma(1-\mu+\eta+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\eta+k)} z^{k-\mu}. \tag{2.1}$$

**Lemma 2.2.** *([5]). Let* $w(z)$ *be an analytic function in the unit disk* $\mathcal{U}$ *with* $w(0) = 0$*, and let* $0 < r < 1$*. If* $|w(z)|$ *attains at* $z_0$ *its maximum value on the circle* $|z| = r$*, then*

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1). \tag{2.2}$$

## 3. Main Results

We begin by proving

**Theorem 3.1.** *Let* $\delta \in \mathbb{R}\backslash\{0\}, p \in \mathbb{N}, 0 \leq \lambda < 1, \mu < 1, \eta > \max(\lambda,\mu)-p-1,$ *and* $a > 0, b \geq 0,$ *such that* $a+2b \leq 1.$ *If a function* $f(z) \in \mathcal{A}_p$ *satisfies the inequality*

$$\Re\left[1 + z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right] \begin{cases} < \dfrac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \\ > \dfrac{a+b}{\delta(1+a)(1-b)} & (\delta < 0) \end{cases} \qquad (z \in \mathcal{U}), \tag{3.1}$$

*then* $f(z) \in \mathcal{M}_\delta(p;\lambda,\mu,\eta)$.

*Proof.* Let $f(z) \in \mathcal{A}_p$, and define a function $w(z)$ by

$$\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^\delta = (p-\mu)^\delta\left(\frac{1+aw(z)}{1-bw(z)}\right) \qquad (z \in \mathcal{U}). \tag{3.2}$$

Then it follows from (2.1) that $w(z)$ is analytic function in $\mathcal{U}$, and $w(0) = 0$.

Differentiation of (3.2) gives

$$\left\{1+z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right\} \tag{3.3}$$
$$=\frac{1}{\delta}\left(\frac{(a+b)zw'(z)}{(1+aw(z))(1-bw(z))}\right)$$
$$=\phi(z) \text{ (say).}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then, applying Lemma 2.2, we can write

$$z_0w'(z_0)=kw(z_0) \qquad (k\geq 1),$$

and $w(z_0)=e^{i\theta}$ $(\theta\in[0,2\pi))$, so that from (3.3) we have

$$\Re\{\phi(z_0)\}=\frac{k(a+b)}{\delta}\Re\left\{\frac{w(z_0)}{(1+aw(z_0))(1-bw(z_0))}\right\}$$
$$=\frac{k}{\delta}\Re\left\{\frac{1}{1-bw(z_0)}-\frac{1}{1+aw(z_0)}\right\}$$
$$=\frac{k}{\delta}\Re\left\{\frac{1-be^{-i\theta}}{1+b^2-2b\cos\theta}-\frac{1+ae^{-i\theta}}{1+a^2+2a\cos\theta}\right\}$$
$$=\frac{k}{\delta}\left\{\frac{1}{2+\frac{b^2-1}{1-b\cos\theta}}-\frac{1}{2+\frac{a^2-1}{1+a\cos\theta}}\right\}=\frac{k\Delta}{\delta},$$

where $\theta \neq \cos^{-1}(-1/a)$ and $\theta \neq \cos^{-1}(-1/b)$.

Simple calculations (under the constraints mentioned with the hypotheses for the parameters $a$ and $b$) yield that $\Delta \geq \frac{(a+b)}{(1+a)(1-b)}$, and since $k \geq 1$, it follows that

$$\Re\{\phi(z_0)\} = \frac{k\Delta}{\delta}\begin{cases} > \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta > 0), \\ < \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta < 0). \end{cases} \tag{3.4}$$

This contradicts our condition (3.1), and we conclude from (3.2) that

$$\begin{aligned} \left| \left( \frac{z J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta}} \right)^{\delta} - (p-\mu)^{\delta} \right| &= (p-\mu)^{\delta} \left| \frac{(a+b)w(z)}{1-bw(z)} \right| \\ &< (p-\mu)^{\delta} \left( \frac{a+b}{1-b} \right) \leq (p-\mu)^{\delta}. \end{aligned}$$

This completes the proof of Theorem 3.1. □

Next we prove

**Theorem 3.2.** *Let* $\delta \in \mathbb{R}\backslash\{0\}$, $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda, \mu) - p - 1$, *and* $a > 0$, $b \geq 0$ *such that* $a + 2b \leq 1$. *If a function* $f(z) \in \mathcal{A}_p$ *satisfies the inequality*

$$\Re\left( \frac{z J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta}} \right) \begin{cases} < p - \mu + \frac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \\ > p - \mu + \frac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \end{cases} \quad (z \in \mathcal{U}), \tag{3.5}$$

*then* $f(z) \in T_{\delta}(p; \lambda, \mu, \eta)$.

*Proof.* Consider

$$(3.6)\quad \left(z^{\mu-p}J_{0,z}^{\lambda,\mu,\eta}f(z)\right)^{\delta} = \left(\frac{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}\right)^{\delta}\left(\frac{1+aw(z)}{1-bw(z)}\right) \quad (z\in\mathcal{U}).$$

Using the same method as elucidated in the proof of Theorem 3.1, we arrive at the desired result. □

**Remark 3.1.** *If we set $\lambda=\mu, a=1, b=0$, then Theorems 3.1 and 3.2 by appealing to the operational relation (1.8) correspond to the recently established results due to Irmak et al. [4, pp. 271–272].*

Theorems 3.1 and 3.2 would also yield various results involving analytic and multivalent functions by suitably choosing the values of $a$, $b$, $\delta$, $\mu$ and $p$. Setting $\delta=1$ in Theorems 3.1 and 3.2, we have

**Corollary 3.3.** *Let $p\in\mathbb{N}$, $0\le\lambda<1$, $\mu<1$, $\eta>\max(\lambda,\mu)-p-1$, and $a>0$, $b\geq 0$ such that $a+2b\leq 1$. If a function $f(z)\in\mathcal{A}_p$ satisfies the inequality*

$$(3.7)\quad \Re\left\{1+z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z\in\mathcal{U}),$$

*then $f(z)\in\mathcal{M}_1(p;\lambda,\mu,\eta)$.*

**Corollary 3.4.** *Let* $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda, \mu) - p - 1$, *and* $a > 0, b \geq 0$ *such that* $a + 2b \leq 1$. *If a function* $f(z) \in \mathcal{A}_p$ *satisfies the inequality*

$$\Re\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right) < p - \mu + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.8}$$

*then* $f(z) \in \mathcal{T}_1(p; \lambda, \mu, \eta)$.

Corollaries 3.3 and 3.4 on putting $\lambda = \mu = 0$, and using (1.8) give the following results:

**Corollary 3.5.** *Let* $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ *such that* $a + 2b \leq 1$. *If a function* $f(z) \in \mathcal{A}_p$ *satisfies the inequality*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.9}$$

*then* $f(z)$ *is* $p$*-valently starlike in* $\mathcal{U}$.

**Corollary 3.6.** *Let* $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ *such that* $a + 2b \leq 1$. *If a function* $f(z) \in \mathcal{A}_p$ *satisfies the inequality*

$$\Re\left\{\frac{zf\prime(z)}{f(z)}\right\} < p + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.10}$$

*then* $\Re\left\{\frac{f(z)}{z^p}\right\} > 0, \quad (z \in \mathcal{U})$.

Lastly, Corollaries 3.3 and 3.4 on putting $\lambda = \mu = 1$, and using (1.8) give

**Corollary 3.7.** *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{1 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)}\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.11}$$

*then $f(z)$ is $p-$valently convex in $\mathcal{U}$.*

**Corollary 3.8.** *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\} < p - 1 + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.12}$$

*then $f(z)$ is $p-$valently close-to -convex in $\mathcal{U}$.*

**Remark 3.2.** *When $a = 1$, $b = 0$, then the Corollaries 3.5 – 3.8 correspond to the known results [3, pp. 457–458] involving inequalities on $p-$valent functions.*

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