

AN IDENTITY IN REAL INNER PRODUCT SPACES

JIANGUO MA

Department of Mathematics
Zhengzhou University
Henan, China
EMail: majg@zzu.edu.cn

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Abstract: We obtain an identity in real inner product spaces that leads to the Grüss inequality and an inequality of Ostrowski.



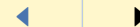
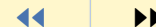
**Identity In Real Inner
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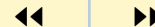
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1. Introduction

The Grüss inequality was generalized by S.S. Dragomir to the inner product spaces in [1]. It turned out to be an inequality relative to the inner products and norms of vectors in inner product space, that is,

“Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $e \in H$, $\|e\| = 1$. if $\phi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the condition

$$(1.1) \quad \operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0, \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

holds, then

$$(1.2) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.”$$

In this paper, we give an identity that yields the inequality

$$(1.3) \quad \left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right|^2 \\ \leq \left[\|x\|^2 - \frac{1}{\|z\|^2} \langle x, z \rangle^2 \right] \left[\|y\|^2 - \frac{1}{\|z\|^2} \langle y, z \rangle^2 \right]$$

here $x, y, z \in H$, H is a real inner product space.

From inequality (1.3), we obtain the Grüss inequality and an inequality by A. Ostrowski.



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2. Main Result

Let x, y, z be three vectors in real inner product spaces. Denote by $Z := \text{span}\{z\}$ the linear subspace spanned by z , and $W := \text{span}\{x, z\}$ the linear subspace spanned by x and z , denote by $\text{dist}(x, \text{span}\{z\}) = \inf_{-\infty < s < +\infty} \|x - sz\|$ for the distance between x and $\text{span}\{z\}$, and $\text{dist}(z, \text{span}\{x, y\}) = \inf_{-\infty < s, t < +\infty} \|z - (sx + ty)\|$. The main result of this paper is:

Theorem 2.1. *Suppose x, y, z are three non-zero vectors in a real inner product space, then*

$$\begin{aligned} \text{dist}^2(x, \text{span}\{z\}) \text{dist}^2(y, \text{span}\{z\}) - \left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right|^2 \\ = \frac{\|y\|^2}{\|z\|^2} \text{dist}^2(x, \text{span}\{y\}) \text{dist}^2(z, \text{span}\{x, y\}). \end{aligned}$$

Proof. Let $D = \text{dist}^2(x, \text{span}\{y\})\|y\|^2$. It is easy to see that

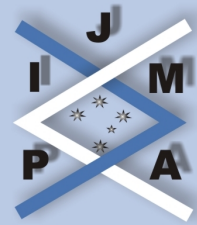
$$(2.1) \quad D = \|x\|^2\|y\|^2 - \langle x, y \rangle^2.$$

When $D \neq 0$, we determine the infimum of $J(s, t) = \|z - (sx + ty)\|^2$ by discovering critical points of $J(s, t)$. Simple calculus yields

$$J(s, t) = \|z\|^2 - 2 \langle x, z \rangle s - 2 \langle y, z \rangle t + \|x\|^2 s^2 + 2 \langle x, y \rangle st + \|y\|^2 t^2,$$

thus partial derivatives of $J(s, t)$ are

$$(2.2) \quad \begin{aligned} \frac{\partial J}{\partial s} &= 2\|x\|^2 s + 2 \langle x, y \rangle t - 2 \langle x, z \rangle \\ \frac{\partial J}{\partial t} &= 2 \langle x, y \rangle s + 2\|y\|^2 t - 2 \langle y, z \rangle. \end{aligned}$$



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Let $\frac{\partial J}{\partial s} = 0$ and $\frac{\partial J}{\partial t} = 0$, we obtain

$$(2.3) \quad \begin{aligned} s &= \frac{1}{D} (\|y\|^2 \langle x, z \rangle - \langle y, z \rangle \langle x, y \rangle) \\ t &= \frac{1}{D} (\|x\|^2 \langle y, z \rangle - \langle x, z \rangle \langle x, y \rangle). \end{aligned}$$

Substituting for s and t in

$$J(s, t) = \|z\|^2 - 2 \langle x, z \rangle s - 2 \langle y, z \rangle t + \|x\|^2 s^2 + 2 \langle x, y \rangle st + \|y\|^2 t^2,$$

by (2.3), we obtain

$$(2.4) \quad \text{dist}^2(z, \text{span}\{x, y\}) = \frac{\|x\|^2 \|y\|^2 \|z\|^2}{D} \\ \times \left(1 - \frac{\langle x, z \rangle^2}{\|x\|^2 \|z\|^2} - \frac{\langle y, z \rangle^2}{\|y\|^2 \|z\|^2} - \frac{\langle x, y \rangle^2}{\|x\|^2 \|y\|^2} + 2 \frac{\langle x, z \rangle \langle y, z \rangle \langle x, y \rangle}{\|x\|^2 \|y\|^2 \|z\|^2} \right).$$

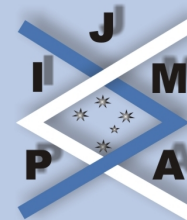
On the other hand, we have

$$(2.5) \quad \text{dist}^2(x, \text{span}\{z\}) \text{dist}^2(y, \text{span}\{z\}) - \left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right|^2 \\ = \left(\|x\|^2 - \frac{\langle x, z \rangle^2}{\|z\|^2} \right) \left(\|y\|^2 - \frac{\langle y, z \rangle^2}{\|z\|^2} \right) - \left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right|^2 \\ = \|x\|^2 \|y\|^2 \left(1 - \frac{\langle x, z \rangle^2}{\|x\|^2 \|z\|^2} - \frac{\langle y, z \rangle^2}{\|y\|^2 \|z\|^2} \right. \\ \left. - \frac{\langle x, y \rangle^2}{\|x\|^2 \|y\|^2} + 2 \frac{\langle x, z \rangle \langle y, z \rangle \langle x, y \rangle}{\|x\|^2 \|y\|^2 \|z\|^2} \right).$$

Comparing (2.4) and (2.5), and taking note that $D = \text{dist}^2(x, \text{span}\{y\})\|y\|^2$, we finish our proof for the case $D \neq 0$.

When $D = 0$, then x and y are linearly dependent. in this case we can prove the theorem by straightforward verification. \square

We point out that Theorem 2.1 is true also for complex inner product spaces.



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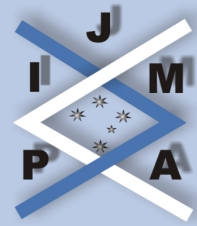
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3. Applications

An application of Theorem 2.1 is the well known Grüss inequality [2] (see also [3]).

Theorem 3.1 (G. Grüss). *Let f and g be two Lebesgue integrable functions on (a, b) . m, M and n, N are four real numbers such that*

$$(3.1) \quad m \leq f(x) \leq M, \quad n \leq g(x) \leq N$$

for each $x \in (a, b)$, then we have the Grüss inequality

$$(3.2) \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(M-m)(N-n).$$

Proof. We consider the Hilbert space $L^2(a, b)$ equipped with an inner product defined by

$$(3.3) \quad \langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x)g(x)dx.$$

According to Theorem 2.1, we have

$$(3.4) \quad \left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right| \leq \text{dist}(x, \text{span}\{z\}) \text{dist}(y, \text{span}\{z\}).$$

This inequality yields inequality (1.3) by (2.1).

Let $x = f$, $y = g$ and $z = 1$. Note that by $m \leq f(x) \leq M$ and $n \leq g(x) \leq N$, it is easy to see that

$$(3.5) \quad \left(f(x) - \frac{m+M}{2} \right)^2 \leq \frac{(M-m)^2}{4}$$



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and

$$(3.6) \quad \left(g(x) - \frac{n+N}{2}\right)^2 \leq \frac{(N-n)^2}{4}.$$

Therefore,

$$(3.7) \quad \text{dist}(f, \text{span}\{1\}) \leq \left(\frac{1}{b-a} \int_a^b (f(x) - \frac{M+m}{2})^2 dx\right)^{\frac{1}{2}} \leq \frac{M-m}{2}.$$

An identical argument yields

$$(3.8) \quad \text{dist}(g, \text{span}\{1\}) \leq \frac{N-n}{2}.$$

Substitute x, y and z in (3.4), and by f, g and 1, we obtain (3.2). \square

Theorem 2.1 also contains a useful inequality of A. Ostrowski [4] (see also [3]).

Theorem 3.2 (Ostrowski). *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two linearly independent vectors. If the vector $x = (x_1, \dots, x_n)$ satisfies*

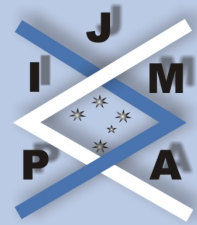
$$(3.9) \quad \sum_{i=1}^n a_i x_i = 0, \quad \sum_{i=1}^n b_i x_i = 1,$$

then

$$(3.10) \quad \sum_{i=1}^n x_i^2 \geq \frac{\sum_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) - (\sum_{i=1}^n a_i b_i)^2}.$$

The equality holds if and only if

$$(3.11) \quad x_k = \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) - (\sum_{i=1}^n a_i b_i)^2}, \quad k = 1, 2, \dots, n.$$



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Proof. Substituting x, y, z in inequality (1.3), by vectors x, a, b , we have

$$(3.12) \quad \left(\|x\|^2 - \frac{1}{\|b\|^2} \right) \left(\|a\|^2 - \frac{\langle a, b \rangle^2}{\|b\|^2} \right) \geq \frac{1}{\|b\|^2} \langle a, b \rangle^2.$$

Simple calculation shows that

$$(3.13) \quad \|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a, b \rangle^2},$$

that is, (3.10). According to Theorem 2.1, equality in (3.13) holds if and only if x, a, b are linearly dependent, that is, there exist constants λ, μ such that $x = \lambda a + \mu b$. Taking the inner product of a and b , we get $\|a\|^2\lambda + \langle a, b \rangle\mu = 0$ and $\langle a, b \rangle\lambda + \|b\|^2\mu = 1$. Solutions of the last two equations are

$$(3.14) \quad \lambda = \frac{-\langle a, b \rangle}{\|a\|^2\|b\|^2 - \langle a, b \rangle^2}, \quad \mu = \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a, b \rangle^2},$$

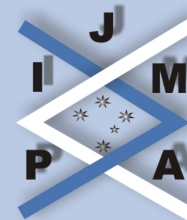
thus

$$(3.15) \quad x = \frac{\|a\|^2 b - \langle a, b \rangle a}{\|a\|^2\|b\|^2 - \langle a, b \rangle^2},$$

that is, (3.11). □

References

- [1] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and application, *J. Math. Anal. Appl.*, **237** (1999), 74–82.
- [2] G. GRÜSS, Über das maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.
- [3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publisher, 1993.
- [4] A. OSTROWSKI, *Vorlesungen Über Differential und Integralrechnung*, Vol. 2, Basel, 1951, p. 289.



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