AN IDENTITY IN REAL INNER PRODUCT SPACES

JIANGUO MA
Department of Mathematics
Zhengzhu University
Henan, China
EMail: majg@zzu.edu.cn

Received: 10 March, 2007

Accepted: 10 May, 2007

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: Primary 26D15; Secondary 46C99.

Key words: Real inner product spaces, Equality, Grüss inequality.

Abstract: We obtain an identity in real inner product spaces that leads to the Grüss inequality and an inequality of Ostrowski.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

vol. 8, iss. 2, art. 48, 2007

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

The Grüss inequality was generalized by S.S. Dragomir to the inner product spaces in [1]. It turned out to be an inequality relative to the inner products and norms of vectors in inner product space, that is,

"Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $e\in H$, $\|e\|=1$. if ϕ,γ,Φ,Γ are real or complex numbers and x,y are vectors in H such that the condition

$$\operatorname{Re}\langle \Phi e-x, x-\phi e\rangle \geq 0,\ \ \operatorname{Re}\langle \Gamma e-y, y-\gamma e\rangle \geq 0 \tag{1.1}$$

holds, then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \leq \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|.\text{"} \tag{1.2}$$

In this paper, we give an identity that yields the inequality

$$\begin{aligned} &\left|\langle x,y\rangle-\frac{1}{\|z\|^2}\langle x,z\rangle\langle y,z\rangle\right|^2 \\ &\qquad \leq \left[\|x\|^2-\frac{1}{\|z\|^2}\langle x,z\rangle^2\right]\left[\|y\|^2-\frac{1}{\|z\|^2}\langle y,z\rangle^2\right] \end{aligned} \tag{1.3}$$

here $x,y,z\in H$, H is a real inner product space.

From inequality (1.3), we obtain the Grüss inequality and an inequality by A. Ostrowski.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Result

Let x, y, z be three vectors in real inner product spaces. Denote by $Z := \text{span}\{z\}$ the linear subspace spanned by z, and $W := \text{span}\{x, z\}$ the linear subspace spanned by x and z, denote by $\text{dist}(x, \text{span}\{z\}) = \inf_{-\infty<s<+\infty} \|x - sz\|$ for the distance between x and $\text{span}\{z\}$, and $\text{dist}(z, \text{span}\{x, y\}) = \inf_{-\infty<s,t<+\infty} \|z - (sx + ty)\|$. The main result of this paper is:

Theorem 2.1. *Suppose x, y, z are three non-zero vectors in a real inner product space, then*

$$\text{dist}^2(x, \text{span}\{z\}) \, \text{dist}^2(y, \text{span}\{z\}) - \left| \langle x, y\rangle - \frac{1}{\|z\|^2} \langle x, z\rangle \langle y, z\rangle \right|^2 = \frac{\|y\|^2}{\|z\|^2} \text{dist}^2(x, \text{span}\{y\}) \, \text{dist}^2(z, \text{span}\{x, y\}).$$

Proof. Let $D = \text{dist}^2(x, \text{span}\{y\})\|y\|^2$. It is easy to see that

$$D = \|x\|^2\|y\|^2 - \langle x, y\rangle^2 . \tag{2.1}$$

When $D \neq 0$, we determine the infimum of $J(s,t) = \|z-(sx+ty)\|^2$ by discovering critical points of $J(s, t)$. Simple calculus yields

$$J(s,t) = \|z\|^2 - 2 \langle x, z\rangle s - 2 \langle y, z\rangle t + \|x\|^2 s^2 + 2 \langle x, y\rangle st + \|y\|^2 t^2,$$

thus partial derivatives of $J(s, t)$ are

$$\frac{\partial J}{\partial s} = 2\|x\|^2 s + 2 \langle x, y\rangle t - 2 \langle x, z\rangle \tag{2.2}$$
$$\frac{\partial J}{\partial t} = 2 \langle x, y\rangle s + 2\|y\|^2 t - 2 \langle y, z\rangle .$$

Identity In Real Inner Product Spaces
Jianguo Ma
vol. 8, iss. 2, art. 48, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Let $\frac{\partial J}{\partial s} = 0$ and $\frac{\partial J}{\partial t} = 0$, we obtain

$$s = \frac{1}{D}(\|y\|^2 \langle x, z\rangle - \langle y, z\rangle \langle x, y\rangle) \tag{2.3}$$
$$t = \frac{1}{D}(\|x\|^2 \langle y, z\rangle - \langle x, z\rangle \langle x, y\rangle).$$

Substituting for s and t in

$$J(s,t) = \|z\|^2 - 2\langle x, z\rangle s - 2\langle y, z\rangle t + \|x\|^2 s^2 + 2\langle x, y\rangle st + \|y\|^2 t^2,$$

by (2.3), we obtain

$$\operatorname{dist}^2(z, \operatorname{span}\{x, y\}) = \frac{\|x\|^2\|y\|^2\|z\|^2}{D} \times \left(1 - \frac{\langle x, z\rangle^2}{\|x\|^2\|z\|^2} - \frac{\langle y, z\rangle^2}{\|y\|^2\|z\|^2} - \frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} + 2\frac{\langle x, z\rangle\langle y, z\rangle\langle x, y\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right). \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} &\operatorname{dist}^2(x, \operatorname{span}\{z\}) \operatorname{dist}^2(y, \operatorname{span}\{z\}) - \left|\langle x, y\rangle - \frac{1}{\|z\|^2}\langle x, z\rangle\langle y, z\rangle\right|^2 \\ &= \left(\|x\|^2 - \frac{\langle x, z\rangle^2}{\|z\|^2}\right)\left(\|y\|^2 - \frac{\langle y, z\rangle^2}{\|z\|^2}\right) - \left|\langle x, y\rangle - \frac{1}{\|z\|^2}\langle x, z\rangle\langle y, z\rangle\right|^2 \\ &= \|x\|^2\|y\|^2\left(1 - \frac{\langle x, z\rangle^2}{\|x\|^2\|z\|^2} - \frac{\langle y, z\rangle^2}{\|y\|^2\|z\|^2}\right. \\ &\qquad \left. - \frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} + 2\frac{\langle x, z\rangle\langle y, z\rangle\langle x, y\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right). \end{aligned} \tag{2.5}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Comparing (2.4) and (2.5), and taking note that $D = \text{dist}^2(x, \text{span}\{y\})\|y\|^2$, we finish our proof for the case $D \neq 0$.

When $D = 0$, then x and y are linearly dependent. in this case we can prove the theorem by straightforward verification. $\square$

We point out that Theorem 2.1 is true also for complex inner product spaces.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Applications

An application of Theorem 2.1 is the well known Grüss inequality [2] (see also [3]).

Theorem 3.1 (G. Grüss). *Let f and g be two Lebesque integrable functions on (a, b). m, M and n, N are four real numbers such that*

$$m \le f(x) \le M, \quad n \le g(x) \le N \tag{3.1}$$

for each $x \in (a, b)$, then we have the Grüss inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \le \frac{1}{4}(M-m)(N-n). \tag{3.2}$$

Proof. We consider the Hilbert space $L^2(a, b)$ equipped with an inner product defined by

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x)g(x)dx. \tag{3.3}$$

According to Theorem 2.1, we have

$$\left| \langle x, y \rangle - \frac{1}{\|z\|^2} \langle x, z \rangle \langle y, z \rangle \right| \le \operatorname{dist}(x, \operatorname{span}\{z\}) \operatorname{dist}(y, \operatorname{span}\{z\}). \tag{3.4}$$

This inequality yields inequality (1.3) by (2.1).

Let $x = f$, $y = g$ and $z = 1$. Note that by $m \le f(x) \le M$ and $n \le g(x) \le N$, it is easy to see that

$$\left(f(x) - \frac{m+M}{2} \right)^2 \le \frac{(M-m)^2}{4} \tag{3.5}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and

$$\left(g(x) - \frac{n+N}{2}\right)^2 \leq \frac{(N-n)^2}{4}. \tag{3.6}$$

Therefore,

$$\operatorname{dist}(f, \operatorname{span}\{1\}) \leq \left(\frac{1}{b-a}\int_a^b (f(x) - \frac{M+m}{2})^2 dx\right)^{\frac{1}{2}} \leq \frac{M-m}{2}. \tag{3.7}$$

An identical argument yields

$$\operatorname{dist}(g, \operatorname{span}\{1\}) \leq \frac{N-n}{2}. \tag{3.8}$$

Substitute x, y and z in (3.4), and by f, g and 1, we obtain (3.2). □

Theorem 2.1 also contains a useful inequality of A. Ostrowski [4] (see also [3]).

Theorem 3.2 (Ostrowski). *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two linearly independent vectors. If the vector $x = (x_1, \ldots, x_n)$ satisfies*

$$\sum_{i=1}^{n} a_i x_i = 0, \quad \sum_{i=1}^{n} b_i x_i = 1, \tag{3.9}$$

then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2}. \tag{3.10}$$

The equality holds if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2}, \quad k = 1, 2, \ldots, n. \tag{3.11}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. Substituting x, y, z in inequality (1.3), by vectors x, a, b, we have

$$\left(\|x\|^2 - \frac{1}{\|b\|^2}\right)\left(\|a\|^2 - \frac{\langle a,b\rangle^2}{\|b\|^2}\right) \geq \frac{1}{\|b\|^2}\langle a,b\rangle^2. \tag{3.12}$$

Simple calculation shows that

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}, \tag{3.13}$$

that is, (3.10). According to Theorem 2.1, equality in (3.13) holds if and only if x, a, b are linearly dependent, that is, there exist constants λ, μ such that $x = \lambda a + \mu b$. Taking the inner product of a and b, we get $\|a\|^2\lambda + \langle a,b\rangle\mu = 0$ and $\langle a,b\rangle\lambda + \|b\|^2\mu = 1$. Solutions of the last two equations are

$$\lambda = \frac{-\langle a,b\rangle}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}, \quad \mu = \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}, \tag{3.14}$$

thus

$$x = \frac{\|a\|^2 b - \langle a,b\rangle a}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}, \tag{3.15}$$

that is, (3.11). □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] S.S. DRAGOMIR, A generalization of Grüss'inequality in inner product spaces and application, *J. Math. Anal. Appl.*, **237** (1999), 74–82.

[2] G. GRÜSS, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publisher, 1993.

[4] A. OSTROWSKI, *Vorlesungen Über Differential und Integralrechnung*, Vol. 2, Basel, 1951, p. 289.

vol. 8, iss. 2, art. 48, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756