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SOME INEQUALITIES FOR KUREPA'S FUNCTION

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper we consider Kurepa's function $K(z)$ [3]. We give some recurrent relations for Kurepa's function via appropriate sequences of rational functions and gamma function. Also, we give some inequalities for Kurepa's function $K(x)$ for positive values of x .

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Contents

1	Kurepa's Function $K(z)$	3
2	Representation of the Kurepa's Function via Sequences of Polynomials and the Gamma Function	4
3	Representation of the Kurepa's Function via Sequences of Rational Functions and the Gamma Function	5
4	Some Inequalities for Kurepa's Function	8
	References	



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 2 of 13

1. Kurepa's Function $K(z)$

Đuro Kurepa considered, in the article [3], the function of left factorial $!n$ as a sum of factorials $!n = 0! + 1! + 2! + \dots + (n-1)!$. Let us use the standard notation:

$$(1.1) \quad K(n) = \sum_{i=0}^{n-1} i!.$$

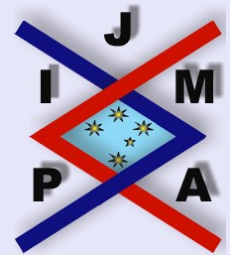
Sum (1.1) corresponds to the sequence A003422 in [5]. Analytical extension of the function (1.1) over the set of complex numbers is determined by the integral:

$$(1.2) \quad K(z) = \int_0^{\infty} e^{-t} \frac{t^z - 1}{t - 1} dt,$$

which converges for $\operatorname{Re} z > 0$ [4]. For function $K(z)$ we use the term *Kurepa's function*. It is easily verified that Kurepa's function $K(z)$ is a solution of the functional equation:

$$(1.3) \quad K(z) - K(z-1) = \Gamma(z).$$

Let us observe that since $K(z-1) = K(z) - \Gamma(z)$, it is possible to make the analytic continuation of Kurepa's function $K(z)$ for $\operatorname{Re} z \leq 0$. In that way, the Kurepa's function $K(z)$ is a meromorphic function with simple poles at $z = -1$ and $z = -n$ ($n \geq 3$) [4]. Let us emphasize that in the following consideration, in Sections 2 and 3, it is sufficient to use only the fact that function $K(z)$ is a solution of the functional equation (1.3).



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 3 of 13

2. Representation of the Kurepa's Function via Sequences of Polynomials and the Gamma Function

Duro Kurepa considered, in article [4], the sequences of following polynomials:

$$(2.1) \quad P_n(z) = (z - n)P_{n-1}(z) + 1,$$

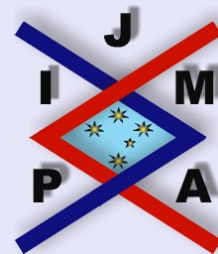
with an initial member $P_0(z) = 1$. On the basis of [4] we can conclude that the following statements are true:

Lemma 2.1. *For each $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have explicitly:*

$$(2.2) \quad P_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^j (z - n + i).$$

Theorem 2.2. *For each $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0, 1, \dots, n\})$ is valid:*

$$(2.3) \quad K(z) = K(z - n) + (P_n(z) - 1) \cdot \Gamma(z - n).$$



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 4 of 13

3. Representation of the Kurepa's Function via Sequences of Rational Functions and the Gamma Function

Let us observe that on the basis of a functional equation for the gamma function $\Gamma(z + 1) = z\Gamma(z)$, it follows that the Kurepa function is the solution of the following functional equation:

$$(3.1) \quad K(z + 1) - (z + 1)K(z) + zK(z - 1) = 0.$$

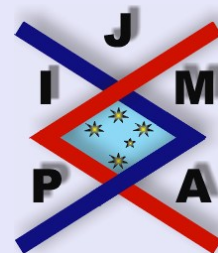
For $z \in \mathbb{C} \setminus \{0\}$, based on (3.1), we have:

$$(3.2) \quad K(z - 1) = \frac{z + 1}{z}K(z) - \frac{1}{z}K(z + 1) = Q_1(z)K(z) - R_1(z)K(z + 1),$$

for rational functions $Q_1(z) = \frac{z+1}{z}$, $R_1(z) = \frac{1}{z}$ over $\mathbb{C} \setminus \{0\}$. Next, for $z \in \mathbb{C} \setminus \{0, 1\}$, based on (3.1), we obtain

$$(3.3) \quad \begin{aligned} K(z - 2) &= \frac{z}{z - 1}K(z - 1) - \frac{1}{z - 1}K(z) \\ &\stackrel{(3.2)}{=} \frac{z}{z - 1} \left(\frac{z + 1}{z}K(z) - \frac{1}{z}K(z + 1) \right) - \frac{1}{z - 1}K(z) \\ &= \frac{z}{z - 1}K(z) - \frac{1}{z - 1}K(z + 1) \\ &= Q_2(z)K(z) - R_2(z)K(z + 1), \end{aligned}$$

for rational functions $Q_2(z) = \frac{z}{z-1}$, $R_2(z) = \frac{1}{z-1}$ over $\mathbb{C} \setminus \{0, 1\}$. Thus, for values $z \in \mathbb{C} \setminus \{0, 1, \dots, n - 1\}$, based on (3.1), by mathematical induction we



have:

$$(3.4) \quad K(z-n) = Q_n(z)K(z) - R_n(z)K(z+1),$$

for rational functions $Q_n(z)$, $R_n(z)$ over $\mathbb{C} \setminus \{0, 1, \dots, n-1\}$, which fulfill the same recurrent relations:

$$(3.5) \quad Q_n(z) = \frac{z-n+2}{z-n+1}Q_{n-1}(z) - \frac{1}{z-n+1}Q_{n-2}(z)$$

and

$$(3.6) \quad R_n(z) = \frac{z-n+2}{z-n+1}R_{n-1}(z) - \frac{1}{z-n+1}R_{n-2}(z),$$

with different initial functions $Q_{1,2}(z)$ and $R_{1,2}(z)$.

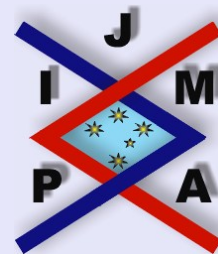
Based on the previous consideration we can conclude:

Lemma 3.1. For each $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0, 1, \dots, n-1\}$ let the rational function $Q_n(z)$ be determined by the recurrent relation (3.5) with initial functions $Q_1(z) = \frac{z+1}{z}$ and $Q_2(z) = \frac{z}{z-1}$. Thus the sequence $Q_n(z)$ has an explicit form:

$$(3.7) \quad Q_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^j \frac{1}{z-i}.$$

Lemma 3.2. For each $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0, 1, \dots, n-1\}$ let the rational function $R_n(z)$ be determined by the recurrent relation (3.6) with initial functions $R_1(z) = \frac{1}{z}$ and $R_2(z) = \frac{1}{z-1}$. Thus the sequence $R_n(z)$ has an explicit form:

$$(3.8) \quad R_n(z) = \sum_{j=0}^{n-1} \prod_{i=0}^j \frac{1}{z-i}.$$



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

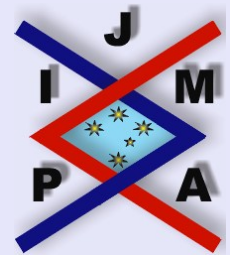
Page 6 of 13

Theorem 3.3. For each $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0, 1, \dots, n-1\})$ we have

$$(3.9) \quad K(z) = K(z-n) + (Q_n(z) - 1) \cdot \Gamma(z+1)$$

and

$$(3.10) \quad K(z) = K(z-n) + R_n(z) \cdot \Gamma(z+1).$$



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 7 of 13

4. Some Inequalities for Kurepa's Function

In this section we consider the Kurepa function $K(x)$, given by an integral representation (1.2), for positive values of x . Thus the Kurepa function is positive and in the following consideration we give some inequalities for the Kurepa function.

Lemma 4.1. For $x \in [0, 1]$ the following inequalities are true:

$$(4.1) \quad \Gamma\left(x + \frac{1}{2}\right) < x^2 - \frac{7}{4}x + \frac{9}{5}$$

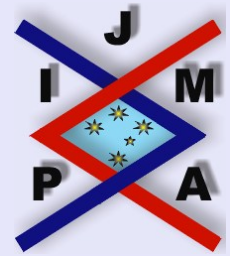
and

$$(4.2) \quad (x + 2)\Gamma(x + 1) > \frac{9}{5}.$$

Proof. It is sufficient to use an approximation formula for the function $\Gamma(x + 1)$ with a polynomial of the fifth degree:

$$P_5(x) = -0.1010678 x^5 + 0.4245549 x^4 - 0.6998588 x^3 \\ + 0.9512363 x^2 - 0.5748646 x + 1$$

which has an absolute error $|\varepsilon(x)| < 5 \cdot 10^{-5}$ for values of argument $x \in [0, 1]$ [1] (formula 6.1.35, page 257). To prove the first inequality, for values $x \in [0, 1/2]$, it is necessary to consider an equivalent inequality obtained by the following substitution $t = x + 1/2$ (thus $\Gamma(x + 1/2) = \Gamma(t + 1)/t$). To prove the first inequality, for values $x \in (1/2, 1]$, it is necessary to consider an equivalent inequality by the following substitution $t = x - 1/2$ (thus $\Gamma(x + 1/2) = \Gamma(t + 1)$). \square



Title Page

Contents



Go Back

Close

Quit

Page 8 of 13

Remark 1. We note that for a proof of the previous inequalities it is possible to use other polynomial approximations (of a lower degree) of functions $\Gamma(x+1/2)$ and $\Gamma(x+1)$ for values $x \in [0, 1]$.

Lemma 4.2. For $x \in [0, 1]$ the following inequality is true:

$$(4.3) \quad K(x) \leq \frac{9}{5}x.$$

Proof. Let us note that the first derivation of Kurepa's function $K(x)$, for values $x \in [0, 1]$, is given by the following integral [4]:

$$(4.4) \quad K'(x) = \int_0^\infty e^{-tx} \frac{\log t}{t-1} dt.$$

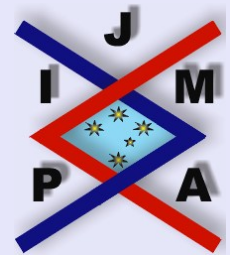
For $t \in (0, \infty) \setminus \{1\}$ Karamata's inequality is true: $\frac{\log t}{t-1} \leq \frac{1}{\sqrt{t}}$ [2]. Hence, for $x \in [0, 1]$ the following inequality is true:

$$(4.5) \quad K'(x) = \int_0^\infty e^{-tx} \frac{\log t}{t-1} dt \leq \int_0^\infty e^{-tx-1/2} dt = \Gamma\left(x + \frac{1}{2}\right).$$

Next, on the basis of Lemma 4.1 and inequality (4.5), for $x \in [0, 1]$, the following inequalities are true:

$$(4.6) \quad K(x) \leq \int_0^x \Gamma\left(t + \frac{1}{2}\right) dt \leq \int_0^x \left(t^2 - \frac{7}{4}t + \frac{9}{5}\right) dt \leq \frac{9}{5}x.$$

□



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 9 of 13

Theorem 4.3. For $x \geq 3$ the following inequality is true:

$$(4.7) \quad K(x-1) \leq \Gamma(x),$$

while the equality is true for $x = 3$.

Proof. Based on the functional equation (1.3) the inequality (4.7), for $x \geq 3$, is equivalent to the following inequality:

$$(4.8) \quad K(x) \leq 2\Gamma(x).$$

Let us represent $[3, \infty) = \bigcup_{n=3}^{\infty} [n, n+1)$. Then, we prove that the inequality (4.8) is true, by mathematical induction over intervals $[n, n+1)$ ($n \geq 3$).

(i) Let $x \in [3, 4)$. Then the following decomposition holds: $K(x) = K(x-3) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x)$. Hence, by Lemma 4.2, the following inequality is true:

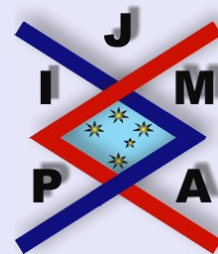
$$(4.9) \quad K(x) \leq \frac{9}{5}(x-3) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x),$$

because $x-3 \in [0, 1)$. Next, by Lemma 4.1, the following inequality is true:

$$(4.10) \quad \frac{9}{5}(x-3) \leq (x-1)(x-3)\Gamma(x-2),$$

because $x-3 \in [0, 1)$. Now, based on (4.9) and (4.10) we conclude that the inequality is true:

$$(4.11) \quad K(x) \leq (x-1)(x-3)\Gamma(x-2) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x) \\ = 2\Gamma(x).$$



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents

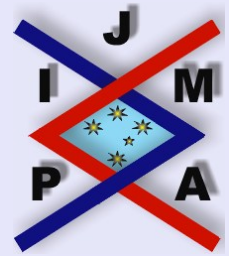


Go Back

Close

Quit

Page 10 of 13



(ii) Let the inequality (4.8) be true for $x \in [n, n + 1)$ ($n \geq 3$).

(iii) For $x \in [n + 1, n + 2)$ ($n \geq 3$), based on the inductive hypothesis, the following inequality is true:

$$(4.12) \quad K(x) = K(x - 1) + \Gamma(x) \leq 2\Gamma(x - 1) + \Gamma(x) \leq 2\Gamma(x).$$

□

Remark 2. The inequality (4.8) is an improvement of the inequalities of Arandjelović: $K(x) \leq 1 + 2\Gamma(x)$, given in [4], with respect to the interval $[3, \infty)$.

Corollary 4.4. For each $k \in \mathbb{N}$ and $x \geq k + 2$ the following inequality is true:

$$(4.13) \quad \frac{K(x - k)}{\Gamma(x - k + 1)} \leq 1,$$

while the equality is true for $x = k + 2$.

Theorem 4.5. For each $k \in \mathbb{N}$ and $x \geq k + 2$ the following double inequality is true:

$$(4.14) \quad R_k(x) < \frac{K(x)}{\Gamma(x + 1)} \leq \frac{P_{k-1}(x) + 1}{P_{k-1}(x)} \cdot R_k(x),$$

while the equality is true for $x = k + 2$.

Proof. For each $k \in \mathbb{N}$ and $x > k$ let us introduce the following function $G_k(x) = \sum_{i=0}^{k-1} \Gamma(x - i)$. Thus, the following relations:

$$(4.15) \quad G_k(x) = \Gamma(x + 1) \cdot R_k(x)$$

and

$$(4.16) \quad G_k(x) = \Gamma(x - k) \cdot (P_k(x) - 1)$$

are true. The inequality $G_k(x) < K(x)$ is true for $x > k$. Hence, based on (4.15), the left inequality in (4.14) is true for all $x \geq k + 2$. On the other hand, based on (4.16) and (4.13), for $x \geq k + 2$, the following inequality is true:

$$(4.17) \quad \begin{aligned} \frac{K(x)}{G_k(x)} &= 1 + \frac{K(x - k)}{G_k(x)} = 1 + \frac{K(x - k)}{\Gamma(x - k)(P_k(x) - 1)} \\ &= 1 + \frac{K(x - k)/\Gamma(x - k + 1)}{P_{k-1}(x)} \leq 1 + \frac{1}{P_{k-1}(x)} \\ &= \frac{P_{k-1}(x) + 1}{P_{k-1}(x)}. \end{aligned}$$

Hence, based on (4.15), the right inequality in (4.14) holds for all $x \geq k + 2$. \square

Corollary 4.6. *If for each $k \in \mathbb{N}$ we mark:*

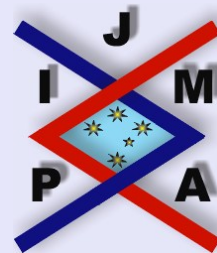
$$(4.18) \quad A_k(x) = R_k(x) \quad \text{and} \quad B_k(x) = \frac{P_{k-1}(x) + 1}{P_{k-1}(x)} \cdot R_k(x),$$

thus, the following is true:

$$(4.19) \quad A_k(x) < A_{k+1}(x) < \frac{K(x)}{\Gamma(x + 1)} \leq B_{k+1}(x) < B_k(x) \quad (x \geq k + 3)$$

and

$$(4.20) \quad A_k(x), B_k(x) \sim \frac{1}{x} \quad \wedge \quad B_k(x) - A_k(x) = \frac{R_k(x)}{P_{k-1}(x)} \sim \frac{1}{x^k} \quad (x \rightarrow \infty).$$



Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

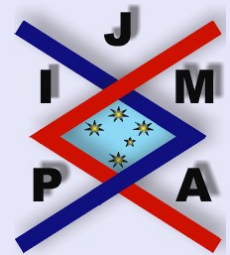
Close

Quit

Page 12 of 13

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Some Inequalities For Kurepa's Function

Branko J. Malešević

Title Page

Contents



Go Back

Close

Quit

Page 13 of 13