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ON HYERS-ULAM STABILITY OF GENERALIZED WILSON'S EQUATION

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Abstract

In this paper, we study the Hyers-Ulam stability problem for the following functional equation

$$(E(K)) \qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi| f(x)g(y), \;\; x,y\in G,$$

where G is a locally compact group, K is a compact subgroup of G, ω_K is the normalized Haar measure of K, Φ is a finite group of K-invariant morphisms of G and $f, g : G \longrightarrow \mathbb{C}$ are continuous complex-valued functions such that f satisfies the Kannappan type condition, for all $x, y, z \in G$

$$(*) \qquad \int_K\int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyk^{-1}hxh^{-1})d\omega_K(k)d\omega_K(h).$$

Our results generalize and extend the Hyers-Ulam stability obtained for the Wilson's functional equation.

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Key words: Functional equations, Hyers-Ulam stability, Wilson equation, Gelfand pairs.

Contents

1. Introduction

Let G be a locally compact group. Let K be a compact subgroup of G. Let ω_K be the normalized Haar measure of K. A mapping $\varphi : G \to G$ is a morphism of G if φ is a homeomorphism of G onto itself which is either a group-homomorphism, i.e. $(\varphi(xy) = \varphi(x)\varphi(y),\ x, y \in G)$, or a group-antihomomorphism, i.e. $(\varphi(xy) = \varphi(y)\varphi(x),\ x, y \in G)$. We denote by $Mor(G)$ the group of morphism of G and Φ a finite subgroup of $Mor(G)$ of a K-invariant morphisms of G (i.e. $\varphi(K) \subset K$). The number of elements of a finite group Φ will be designated by $|\Phi|$. The Banach algebra of bounded measures on G with complex values is denoted by $M(G)$ and the Banach space of all complex measurable and essentially bounded functions on G by $L_\infty(G)$. $\mathcal{C}(G)$ designates the Banach space of all continuous complex valued functions on G.

In this paper we are going to generalize the results obtained in [1], [4] and [6] for the integral equation

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|f(x)g(y), \qquad x, y \in G. \tag{1.1}$$

This equation may be considered as a common generalization of functional equations of Cauchy and Wilson type

$$f(xy) = f(x)g(y), \qquad x, y \in G, \tag{1.2}$$

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \qquad x, y \in G, \tag{1.3}$$

where σ is an involution of G. It is also a generalization of the equations

$$\int_K f(xkyk^{-1})d\omega_K(k) = f(x)g(y), \qquad x, y \in G, \tag{1.4}$$

$$\int_K f(xkyk^{-1})d\omega_K(k) + \int_K f(xk\sigma(y)k^{-1})d\omega_K(k) = 2f(x)g(y), \qquad x, y \in G, \tag{1.5}$$

$$\int_K f(xky)\overline{\chi}(k)d\omega_K(k) = f(x)g(y), \qquad x, y \in G, \tag{1.6}$$

$$\int_K f(xky)\overline{\chi}(k)d\omega_K(k) + \int_K f(xk\sigma(y))\overline{\chi}(k)d\omega_K(k) = 2f(x)g(y), \qquad x, y \in G, \tag{1.7}$$

$$\int_K f(xky)d\omega_K(k) = f(x)g(y), \qquad x, y \in G, \tag{1.8}$$

and

$$\int_K f(xky)d\omega_K(k) + \int_K f(xk\sigma(y))d\omega_K(k) = 2f(x)g(y), \quad x, y \in G. \tag{1.9}$$

If G is a compact group, the equation (1.1) may be considered as a generalization of the equations

$$\int_G f(xtyt^{-1})dt = f(x)g(y), \qquad x, y \in G, \tag{1.10}$$

$$\int_G f(xtyt^{-1})dt + \int_G f(xt\sigma(y)t^{-1})dt = 2f(x)g(y), \qquad x, y \in G, \tag{1.11}$$

and

$$\sum_{\varphi\in\Phi} \int_G f(xt\varphi(y)t^{-1})dt = |\Phi| f(x)g(y), \qquad x, y \in G. \tag{1.12}$$

Furthermore the following equations are also a particular case of (1.1).

$$\sum_{\varphi\in\Phi} f(x\varphi(y)) = |\Phi| f(x)g(y), \qquad x, y \in G, \tag{1.13}$$

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))d\omega_K(k) = |\Phi| f(x)g(y), \qquad x, y \in G, \tag{1.14}$$

and

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k) = |\Phi| f(x)g(y), \qquad x, y \in G, \tag{1.15}$$

where χ is a unitary character of K.

In the next section, we note some results for later use.

2. General Properties

In what follows, we study general properties. Let G, K and Φ given as above

Proposition 2.1 ([4]). *For an arbitrary fixed $\tau \in \Phi$, the mapping*

$$\begin{aligned} \Phi &\longrightarrow \Phi \\ \varphi &\mapsto \varphi \circ \tau \end{aligned}$$

is a bijection and for all $x, y \in G$, we have

$$\sum_{\varphi \in \Phi} \int_K f(xk\varphi(\tau(y))k^{-1}) d\omega_K(k) = \sum_{\psi \in \Phi} \int_K f(xk\psi(y)k^{-1}) d\omega_K(k). \tag{2.1}$$

Proposition 2.2. *Let $\varphi \in \Phi$ and $f \in \mathcal{C}(G)$, then we have*

i)

$$\int_K f(xk\varphi(hy)k^{-1}) d\omega_K(k) = \int_K f(xk\varphi(yh)k^{-1}) d\omega_K(k), \ \ x, y \in G, h \in K.$$

ii) If f satisfies (*), then for all $a, z, y, x \in G$, we have

$$\begin{aligned} &\int_K \int_K f(zh\varphi(ykxk^{-1})h^{-1}) d\omega_K(h) d\omega_K(k) \\ &\qquad = \int_K \int_K f(zh\varphi(xkyk^{-1})h^{-1}) d\omega_K(h) d\omega_K(k). \end{aligned}$$

and

$$\int_K \int_K \int_K f(ah\varphi(zk_1yk_1^{-1}h_1xh_1^{-1})h^{-1})d\omega_K(h)d\omega_K(k_1)d\omega_K(h_1)$$
$$= \int_K \int_K \int_K f(ah\varphi(zk_1xk_1^{-1}h_1yh_1^{-1})h^{-1})d\omega_K(h)d\omega_K(k_1)d\omega_K(h_1).$$

Proof. By easy computations. □

3. The Main Results

The main result is the following theorem.

Theorem 3.1. *Let* $\delta > 0$ *and let* $(f, g) \in \mathcal{C}(G)$ *such that* f *satisfies the condition (*) and the functional inequality*

$$\left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi| f(x)g(y) \right| \leq \delta, \qquad x, y \in G. \tag{3.1}$$

Then

i) f, g *are bounded or*

ii) f *is unbounded and* g *satisfies the equation*

$$\sum_{\varphi \in \Phi} \int_K g(xk\varphi(y)k^{-1}) d\omega_K(k) = |\Phi| g(x)g(y), \qquad x, y \in G. \tag{3.2}$$

iii) g *is unbounded,* f *satisfies the equation (1.1). Furthermore if* $f \neq 0$, *then* g *is a solution of (3.2).*

Proof. Let (f, g) be a solution of the inequality (3.1), such that f is unbounded and satisfies the condition (*), then for all $x, y, z \in G$, we get by using Propositions 2.1 and 2.2

$$|\Phi||f(z)| \left| \sum_{\varphi \in \Phi} \int_K g(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi| g(x)g(y) \right|$$

$$= \left| \sum_{\varphi\in\Phi} \int_K |\Phi| f(z) g(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi|^2 f(z) g(x) g(y) \right|$$

$$\leq \left| \sum_{\varphi\in\Phi} \int_K \sum_{\psi\in\Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) d\omega_K(k) \right.$$

$$\left. - |\Phi| f(z) \sum_{\varphi\in\Phi} \int_K g(xk\varphi(y)k^{-1}) d\omega_K(k) \right|$$

$$+ \left| \sum_{\psi\in\Phi} \int_K \sum_{\varphi\in\Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) d\omega_K(k) \right.$$

$$\left. - |\Phi| g(y) \sum_{\psi\in\Phi} \int_K f(zk\psi(x)k^{-1}) d\omega_K(k) \right|$$

$$+ |\Phi||g(y)| \left| \sum_{\psi\in\Phi} \int_K f(zh\psi(x)h^{-1}) d\omega_K(h) - |\Phi| f(z) g(x) \right|$$

$$\leq \sum_{\varphi\in\Phi} \int_K \left| \sum_{\psi\in\Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) - |\Phi| f(z) g(xk\varphi(y)k^{-1}) \right| d\omega_K(k)$$

$$+ \sum_{\psi\in\Phi} \int_K \left| \sum_{\tau\in\Phi} \int_K f(zh\psi(x)h^{-1}k\tau(y)k^{-1})) d\omega_K(k) - |\Phi| g(y) f(zh\psi(x)h^{-1}) \right| d\omega_K(h)$$

$$+ |\Phi||g(y)| \left| \sum_{\psi\in\Phi} \int_K f(zk\psi(x)k^{-1}) d\omega_K(k) - |\Phi| f(z) g(x) \right|$$

$$\leq 2|\Phi|\delta + |\Phi||g(y)|\delta.$$

Since f is unbounded it follows that g is a solution of the functional equation (3.2). For the second case let (f, g) be a solution of the inequality (3.1) such that f satisfies the condition (*) and g is unbounded then for all $x, y, z \in G$, one has

$$\begin{aligned}
|\Phi||g(z)| & \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)g(y)\right| \\
&= \left|\sum_{\varphi\in\Phi}\int_K |\Phi|g(z)f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|^2 g(z)f(x)g(y)\right| \\
&\leq \left|\sum_{\psi\in\Phi}\int_K \sum_{\varphi\in\Phi}\int_K f(xh\varphi(y)h^{-1}k\psi(z)k^{-1})d\omega_K(h)d\omega_K(k)\right. \\
&\qquad \left. -|\Phi|g(z)\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k)\right| \\
&\qquad + \left|\sum_{\varphi\in\Phi}\int_K \sum_{\psi\in\Phi}\int_K f(xh\psi(z)h^{-1}k\varphi(y)k^{-1})d\omega_K(h)d\omega_K(k)\right. \\
&\qquad \left. - |\Phi|g(y)\sum_{\psi\in\Phi}\int_K f(xk\psi(z)k^{-1})d\omega_K(k)\right| \\
&\qquad + |\Phi||g(y)|\left|\sum_{\psi\in\Phi}\int_K f(xk\psi(z)k^{-1})d\omega_K(k) - |\Phi|f(x)g(z)\right|
\end{aligned}$$

$$\leq \sum_{\varphi\in\Phi}\int_K \left|\sum_{\psi\in\Phi}\int_K f(xk\varphi(y)k^{-1}h\psi(z)h^{-1})d\omega_K(h)\right.$$
$$\left. - |\Phi|g(z)f(xk\varphi(y)k^{-1})\right|d\omega_K(k)$$
$$+ \sum_{\psi\in\Phi}\int_K \left|\sum_{\varphi\in\Phi}\int_K f(xk\psi(z)k^{-1}h\varphi(y)h^{-1}))d\omega_K(h)\right.$$
$$\left. - |\Phi|g(y)f(xk\psi(z)k^{-1})\right|d\omega_K(k)$$
$$+ |\Phi||g(y)|\left|\sum_{\psi\in\Phi}\int_K f\left(xk\psi(z)k^{-1}\right)d\omega_K(k) - |\Phi|f(x)g(z)\right|$$
$$\leq 2|\Phi|\delta + |\Phi||g(y)|\delta.$$

Since g is unbounded it follows that f is a solution of (1.1). Now let $f \neq 0$, then there exists $a \in G$ such that $f(a) \neq 0$. Let $\eta = \frac{\delta}{|f(a)|}$ and let

$$F(x) = \frac{1}{|\Phi||f(a)|}\sum_{\varphi\in\Phi}\int_K f(ak\varphi(x)k^{-1})d\omega_K(k).$$

By using Proposition 2.2 it follows that F satisfies the condition (*), and by using the inequality (3.1) one has $|F(x) - g(x)| \leq \frac{\eta}{|\Phi|}$, since g is unbounded it follows that F is unbounded. Furthermore for all $x, y \in G$ we have

$$\left|\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|F(x)g(y)\right|$$

$$= \frac{1}{|\Phi|f(a)}\left|\sum_{\varphi\in\Phi}\int_K \Sigma_{\psi\in\Phi}\int_K f(ah\psi(xk\varphi(y)k^{-1})h^{-1})d\omega_K(h)d\omega_K(k)\right.$$

$$\left. - |\Phi|\frac{1}{|\Phi|f(a)}\sum_{\varphi\in\Phi}\int_K f(ak\varphi(x)k^{-1})d\omega_K(k)g(y)\right|$$

$$\leq \frac{1}{|\Phi|f(a)}\sum_{\varphi\in\Phi}\int_K\left|\sum_{\tau\in\Phi}\int_K f(ah\psi(x)h^{-1}k\tau(y)k^{-1})d\omega_K(k)\right.$$

$$\left. - |\Phi|f(ah\varphi(x)h^{-1})g(y)\right|d\omega_K(k)$$

$$\leq \eta.$$

From the first case it follows that g is a solution of (3.2). □

Corollary 3.2. *Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality*

$$\left|\int_K f(xkyk^{-1})d\omega_K(k)\right.$$
$$\left. + \int_K f(xk\sigma(y)k^{-1})d\omega_K(k) - 2f(x)g(y)\right| \leq \delta, \quad x, y \in G, \tag{3.3}$$

where σ is an involution on G. Then

i) *f, g are bounded or*

ii) *f is unbounded and g satisfies the equation*

$$\int_K g(xkyk^{-1})d\omega_K(k) + \int_K g(xk\sigma(y)k^{-1})d\omega_K(k) = 2g(x)g(y), \qquad x, y \in G. \tag{3.4}$$

iii) *g is unbounded, f satisfies the equation (1.5). Furthermore if $f \neq 0$, then g is a solution of (3.4).*

Remark 3.1. *In the case where $\Phi = \{I\}$, it is not necessary to assume that f satisfies the condition (*) (see [1] and [6]).*

4. Applications

The following theorems are a particular case of Theorem 3.1.

If $K \subset Z(G)$, then we have

Theorem 4.1. *Let $\delta > 0$ and let f, g be a complex-valued functions on G such that f satisfies the Kannappan condition ([12])*

$$f(zxy) = f(zyx), \qquad x, y \in G \tag{*}$$

and the functional inequality

$$\left| \sum_{\varphi \in \Phi} f(x\varphi(y)) - |\Phi| f(x)g(y) \right| \leq \delta, \qquad x, y \in G. \tag{4.1}$$

Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$\sum_{\varphi \in \Phi} g(x\varphi(y)) = |\Phi| g(x)g(y), \qquad x, y \in G, \tag{4.2}$$

iii) g is unbounded and f is a solution of (1.13). Furthermore if $f \neq 0$ then g is a solution of (4.2).

Corollary 4.2. *Let $\delta > 0$ and let f, g be a complex-valued functions on G such that f satisfies the Kannappan condition*

$$(*) \qquad f(zxy) = f(zyx), \qquad x, y \in G$$

and the functional inequality

$$(4.3) \qquad |f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad x, y \in G,$$

where σ is an involution on G. Then

i) *f, g are bounded or*

ii) *f is unbounded and g is a solution of the functional equation*

$$(4.4) \qquad g(xy) + g(x\sigma(y)) = 2g(x)g(y), \qquad x, y \in G,$$

iii) *g is unbounded and f is a solution of (1.3). Furthermore if $f \neq 0$ then g is a solution of (4.4).*

Remark 4.1. *If G is abelian, then the condition (*) holds.*

If $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$ and $x \in G$, where χ is a character of K ([13]), then we have

Theorem 4.3. *Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$,*

$$(*) \quad \int_K \int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)$$

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and

$$(4.5)\qquad \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k)-|\Phi|f(x)g(y)\right|\leq\delta,\qquad x,y\in G.$$

Then

i) f, g *are bounded or*

ii) f *is unbounded and* g *is a solution of the functional equation*

$$(4.6)\qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k)=|\Phi|f(x)f(y),\qquad x,y\in G,$$

iii) g *is unbounded and* f *is a solution of (1.15). Furthermore if* $f\neq 0$ *then* g *is a solution of (4.6).*

Corollary 4.4. *Let* $\delta>0$ *and let* $(f,g)\in\mathcal{C}(G)$ *such that* $f(kxh)=\chi(k)f(x)\chi(h)$, $k,h\in K$, $x\in G$,

$$(*)\qquad \int_K\int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)=\int_K\int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)$$

and

$$(4.7)\qquad \left|\int_K f(xky)\overline{\chi}(k)d\omega_K(k)+\int_K f(xk\sigma(y))\overline{\chi}(k)d\omega_K(k)-2f(x)g(y)\right|\leq\delta,\quad x,y\in G.$$

where σ is an involution of G. Then

i) *f, g are bounded or*

ii) *f is unbounded and g is a solution of the functional equation*

$$\int_K g(xky)\overline{\chi}(k)d\omega_K(k) + \int_K g(xk\sigma(y))\overline{\chi}(k)d\omega_K(k) = 2g(x)g(y), \qquad x, y \in G. \tag{4.8}$$

iii) *g is unbounded and f is a solution of (1.7). Furthermore if $f \neq 0$ then g is a solution of (4.8).*

Remark 4.2. *If the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative then the condition (*) holds [4].*

In the next theorem we assume that f to be bi-K-invariant (i.e. $f(hxk) = f(x), h, k \in K, x \in G$ ([7], [10]), then we have

Theorem 4.5. *Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = f(x)$, $k, h \in K$, $x \in G$,*

$$\int_K \int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)d\omega_K(k)d\omega_K(h), \qquad x, y, z \in G \tag{*}$$

and

$$(4.9)\qquad \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))d\omega_K(k) - |\Phi| f(x)g(y)\right| \le \delta, \qquad x,y\in G.$$

Then

i) f, g *are bounded or*

ii) f *is unbounded and* g *is a solution of the functional equation*

$$(4.10)\qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))d\omega_K(k) = |\Phi| f(x)f(y), \qquad x,y\in G,$$

iii) g *is unbounded and* f *is a solution of (1.14). Furthermore if* $f \neq 0$ *then* g *is a solution of (4.10).*

Corollary 4.6 ([6]). *Let* $\delta > 0$ *and let* $(f,g)\in\mathcal{C}(G)$ *such that* $f(kxh) = f(x)$, $k,h\in K$, $x\in G$,

$$(*)\qquad \int_K\int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyhx)d\omega_K(k)d\omega_K(h), \qquad x,y,z\in G$$

and

$$(4.11)\qquad \left|\int_K f(xky)d\omega_K(k) + \int_K f(xk\sigma(y))d\omega_K(k) - 2f(x)g(y)\right| \le \delta, \qquad x,y\in G.$$

where σ is an involution of G. Then

i) *f, g are bounded or*

ii) *f is unbounded and g is a solution of the functional equation*

$$\int_K g(xky)d\omega_K(k) + \int_K g(xk\sigma(y))d\omega_K(k) = 2g(x)g(y), \qquad x, y \in G. \tag{4.12}$$

iii) *g is unbounded and f is a solution of (1.9). Furthermore if $f \neq 0$ then g is a solution of (4.12).*

Remark 4.3. *If the algebra $\omega_K \star M(G) \star \omega_K$ is commutative then the condition (*) holds [4].*

In the next corollary, we assume that $G = K$ is a compact group

Theorem 4.7. *Let $\delta > 0$ and let (f, g) be complex measurable and essentially bounded functions on G such that f is a central function and (f, g) satisfy the inequality*

$$\left| \sum_{\varphi \in \Phi} \int_G f(xt\varphi(y)t^{-1})dt - |\Phi| f(x)g(y) \right| \leq \delta, \qquad x, y \in G. \tag{4.13}$$

Then

i) *f and g are bounded or*

ii) f is unbounded and g is a solution of the functional equation

$$\sum_{\varphi\in\Phi}\int_G g(xt\varphi(y)t^{-1})dt = |\Phi|g(x)g(y), \qquad x, y \in G. \tag{4.14}$$

iii) g is unbounded and $f \equiv 0$.

Proof. Let $(f, g) \in L^\infty(G)$. Since f is central [5], then it satisfies the condition (*) ([4]). For (iii), if $f \neq 0$ then g is a solution of the functional equation (4.14). In view of the proposition in [9] we get the fact that g is continuous. Since G is compact then g is bounded. □

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