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THE GENERALIZED SINE LAW AND SOME INEQUALITIES FOR SIMPLICES

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Abstract

The sines of k-dimensional vertex angles of an n-simplex is defined and the law of sines for k-dimensional vertex angles of an n-simplex is established. Using the generalized sine law for n-simplex, we obtain some inequalities for the sines of k-dimensional vertex angles of an n-simplex. Besides, we obtain inequalities for volumes of n-simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P. Chiakuei inequality, and an inequality for pedal simplex are given.

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Contents

1. Introduction

The law of sines for triangles in E^2 has natural analogues in higher dimensions. In 1978, F. Eriksson [1] defined the n-dimensional sines of the n-dimensional corners of an n-simplex in n-dimensional Euclidean space E^n and obtained the law of sines for the n-dimensional corners of an n-simplex. In this paper, the sines of k-dimensional vertex angles of an n -simplex will be defined, and the law of sines for k-dimensional vertex angles of an n-simplex will be established. Using the generalized sine law for simplices and a known inequality in [2], we get some inequalities for the sines of k-dimensional vertex angles of an n-simplex.

Recently, Yang Lu and Zhang Jingzhong [2, 3], Yang Shiguo [4], Leng Gangson [5, 6] and D. Veljan [7] and V. Volenec et al. [9] have obtained some important inequalities for volumes of n-simplices. In this paper, some interesting new inequalities for volumes of n-simplices will be established. As corollaries, we will obtain an inequality for pedal simplex and a generalization to several dimensions of the Neuberg-Pedoe inequality, which differs from the results in [4], [5] and [6].

2. The Generalized Sine Law for Simplices

Let A_i $(i = 1, 2, \ldots, n+1)$ be the vertices of an n-dimensional simplex Ω_n in the n-dimensional Euclidean space E^n, V the volume of the simplex Ω_n and $F_i(n-1)$-dimensional content of Ω_n. F. Eriksson defined the n-dimensional sines of the n-dimensional corners α_i of the n-simplex Ω_n and obtained the law of sines for n-simplices as follows [1]

$$\frac{F_i}{{}^n\sin\alpha_i} = \frac{(n-1)!\prod_{j=1}^{n+1} F_j}{(nV)^{n-1}} \qquad (i = 1, 2, \ldots, n+1). \tag{2.1}$$

In this paper, we will define the sines of the k-dimensional vertex angles of an n-dimensional simplex and establish the law of sines for the k-dimensional vertex angles of an n-simplex. Let $V_{i_1 i_2 \cdots i_k}$ be the $(k-1)$-dimensional content of the $(k-1)$-dimensional face $A_{i_1} A_{i_2} \cdots A_{i_k}$ ($(k-1)$-simplex) of the simplex Ω_n for $k \in \{2, 3, \ldots, n+1\}$ and $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n+1\}$, O and R denote the circumcenter and circumradius of the simplex Ω_n respectively. $\overrightarrow{OA_i} = R\boldsymbol{u_i} (i = 1, 2, \ldots, n+1)$, $\boldsymbol{u_i}$ is the unit vector. The sines of the k-dimensional vertex angles of the simplex Ω_n are defined as follows.

Definition 2.1. *Let α_{ij} denote the angle formed by the vectors $\boldsymbol{u_i}$ and $\boldsymbol{u_j}$. The sine of a k-dimensional vertex angle $\varphi_{i_1 i_2 \cdots i_k}$ of the simplex Ω_n corresponding the $(k-1)$-dimensional face $A_{i_1} A_{i_2} \cdots A_{i_k}$ is defined as*

$$\sin\varphi_{i_1 i_2 \cdots i_k} = (-D_{i_1 i_2 \cdots i_k})^{\frac{1}{2}}, \tag{2.2}$$

where

$$(2.3)\qquad D_{i_1i_2\cdots i_k}=\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{2}\sin^2\frac{\alpha_{i_li_m}}{2}} & \\ 1 & & & \end{vmatrix}\qquad (l,m=1,2,\dots,k).$$

We will prove that

$$(2.4)\qquad 0<(-D_{i_1i_2\cdots i_k})^{\frac{1}{2}}\le 1.$$

If $n=2$, the sine of the 2-dimensional vertex angle φ_{ij} of the triangle $A_1A_2A_3$ is the sine of the angle formed by two edges A_kA_i and A_kA_j.

With the notation introduced above, we establish the law of sines for the k -dimensional vertex angles of an n-simplex as follows.

Theorem 2.1. *For an n-dimensional simplex Ω_n in E^n and $k\in\{2,3,\dots,n+1\}$, we have*

$$(2.5)\qquad \frac{V_{i_1i_2\cdots i_k}}{\sin\varphi_{i_1i_2\cdots i_k}}=\frac{(2R)^{k-1}}{(k-1)!}\qquad (1\le i_1<i_2<\cdots<i_k\le n+1).$$

Put $\varphi_{12\cdots i-1,i+1,\dots,n+1}=\theta_i,\ V_{12\cdots i-1,i+1,\dots,n+1}=F_i\ (i=1,2,\dots,n+1)$, by Theorem 2.1 we obtain the law of sines for the n-dimensional vertex angles of n-simplices as follows.

Corollary 2.2.

$$(2.6)\qquad \frac{F_1}{\sin\theta_1}=\frac{F_2}{\sin\theta_2}=\cdots=\frac{F_{n+1}}{\sin\theta_{n+1}}=\frac{(2R)^{n-1}}{(n-1)!}.$$

If we take $n = 2$ in Theorem 2.1 or Corollary 2.2, we obtain the law of sines for a triangle $A_1A_2A_3$ in the form

$$\frac{a_1}{\sin A_1} = \frac{a_2}{\sin A_2} = \frac{a_3}{\sin A_3} = 2R. \tag{2.7}$$

Proof of Theorem 2.1. Let $a_{ij} = |A_iA_j|$ $(i, j = 1, 2, \dots, n+1)$, then

$$a_{ij} = 2R \sin \frac{\alpha_{ij}}{2},$$

$$\begin{aligned} \sin^2 \varphi_{i_1 i_2 \cdots i_k} &= -D_{i_1 i_2 \cdots i_k} = -\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{8R^2} a_{i_l i_m}^2} & \\ 1 & & & \end{vmatrix} \\ &= (-1)^k (8R^2)^{-(k-1)} \cdot \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{a_{i_l i_m}^2} & \\ 1 & & & \end{vmatrix} \qquad (l, m = 1, 2, \dots, k). \end{aligned} \tag{2.8}$$

By the formula for the volume of a simplex, we have

$$\begin{aligned} \sin^2 \varphi_{i_1 i_2 \cdots i_k} &= -D_{i_1 i_2 \cdots i_k} \\ &= (-1)^k (8R^2)^{-(k-1)} (-1)^k 2^{k-1} (k-1)!^2 V_{i_1 i_2 \cdots i_k}^2 \\ &= \frac{(k-1)!^2}{(2R)^{2(k-1)}} V_{i_1 i_2 \cdots i_k}^2. \end{aligned} \tag{2.9}$$

From this equality (2.5) follows.

Now we prove that inequality (2.4) holds. When $k \geq 2$, we have $k \leq 2^{k-1}$. Using the Voljan-Korchmaros inequality [3], we have

$$V_{i_1 i_2 \cdots i_k} \leq \frac{1}{(k-1)!}\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}}\left(\prod_{1\leq l<m\leq k} a_{i_l i_m}\right)^{\frac{2}{k}}. \tag{2.10}$$

Equality holds if and only if the simplex $A_{i_1}A_{i_2}\cdots A_{i_k}$ is regular.

Combining inequality (2.10) with equality (2.5), we get

$$\begin{aligned} V_{i_1 i_2 \cdots i_k} &\leq \frac{(2R)^{k-1}}{(k-1)!}\cdot\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}}\left(\prod_{1\leq l<m\leq k} \sin\frac{\alpha_{i_l i_m}}{2}\right)^{\frac{2}{k}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!}\cdot\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!}. \end{aligned} \tag{2.11}$$

Using equality (2.5) and inequality (2.11), we get

$$0 < (-D_{i_1 i_2 \cdots i_k})^{\frac{1}{2}} = \sin\varphi_{i_1 i_2 \cdots i_k} = \frac{(k-1)!}{(2R)^{k-1}} V_{i_1 i_2 \cdots i_k} \leq 1.$$

□

For the sines of the k-dimensional vertex angles of an n-simplex, we obtain an inequality as follows.

Theorem 2.3. *Let* $\varphi_{i_1 i_2 \cdots i_k}$ $(1 \le i_1 < i_2 < \cdots < i_k \le n+1)$ *denote the* k*-dimensional vertex angles of an* n*-simplex* Ω_n *in* E^n*, and* $\lambda_i > 0$ $(i = 1, 2, \ldots, n+1)$ *be arbitrary real numbers, then we have*

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n+1} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \sin^2 \varphi_{i_1 i_2 \cdots i_k} \le \frac{n! \cdot \left(\sum_{i=1}^{n+1} \lambda_i\right)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}. \tag{2.12}$$

Equality holds if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ *and the simplex* Ω_n *is regular.*

By taking $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ in the inequality (2.12), we get:

Corollary 2.4.

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n+1} \sin^2 \varphi_{i_1 i_2 \cdots i_k} \le \frac{n! \cdot (n+1)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}. \tag{2.13}$$

Equality holds if the simplex Ω_n *is regular.*

To prove Theorem 2.3, we need a lemma as follows.

Lemma 2.5. *Let* Ω_n *be an* n*-simplex in* E^n, $x_i > 0$ $(i = 1, 2, \ldots, n+1)$ *be real numbers,* $V_{i_1 i_2 \cdots i_{s+1}}$ *be the* s*-dimensional volume of the* s*-dimensional simplex* $A_{i_1} A_{i_2} \cdots A_{i_{s+1}}$ *for* $i_1, i_2, \ldots, i_{s+1} \in \{1, 2, \ldots, n+1\}$*. Put*

$$M_s = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n+1} x_{i_1} x_{i_2} \cdots x_{i_{s+1}} V^2_{i_1 i_2 \cdots i_{s+1}}, \; M_0 = \sum_{i=1}^{n+1} x_i,$$

then we have

$$M_s^l \geq \frac{[(n-l)!(l!)^3]^s}{[(n-s)!(s!)^3]^l}(n!\cdot M_0)^{l-s}M_l^s (1 \leq s < l \leq n). \tag{2.14}$$

Equality holds if and only if the intertial ellipsoid of the points $A_1, A_2, \dots, A_{n+1}$ with masses $x_1, x_2, \dots, x_{n+1}$ is a sphere.

For the proof of Lemma 2.5. the reader is referred to [2] or [9].

Proof of Theorem 2.3. By putting $s = 1, l = k-1$ and $x_i = \lambda_i$ $(i = 1, 2, \dots, n+1)$ in the inequality (2.14), we have

$$\begin{aligned}\left(\sum_{1\leq i<j\leq n+1} \lambda_i\lambda_j a_{ij}^2\right)^{k-1} &\geq \frac{(n-k+1)!\cdot(k-1)!^3}{[(n-1)!]^{k-1}}\left(n!\cdot\sum_{i=1}^{n+1}\lambda_i\right)^{k-2} \\ &\quad\times \sum_{1\leq i_1<i_2<\cdots<i_k\leq n+1} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}V_{i_1i_2\cdots i_k}^2.\end{aligned} \tag{2.15}$$

By Theorem 2.1, we have

$$V_{i_1i_2\cdots i_k} = \frac{(2R)^{k-1}}{(k-1)!}\sin\varphi_{i_1i_2\cdots i_k}. \tag{2.16}$$

Using the known inequality [3]

$$\sum_{1\leq i<j\leq n+1} \lambda_i\lambda_j a_{ij}^2 \leq \left(\sum_{i=1}^{n+1}\lambda_i\right)^2 R^2, \tag{2.17}$$

with equality if and only if the point $P = \sum_{i=1}^{n+1} \lambda_i A_i$ is the circumcenter of simplex Ω_n.

Combining (2.15) with (2.16) and (2.17), we obtain inequality (2.12). It is easy to see that equality holds in (2.12) if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ and simplex Ω_n is regular. $\square$

3. Some Inequalities for Volumes of Simplices

Let P be an arbitrary point inside the simplex Ω_n and B_i the orthogonal projection of the point P on the $(n-1)$-dimensional plane σ_i containing $(n-1)$-simplex $f_i = A_1 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$. Simplex $\overline{\Omega}_n = B_1B_2 \cdots B_{n+1}$ is called the pedal simplex of the point P with respect to the simplex Ω_n. Let $r_i = |PB_i|$ $(i = 1, 2, \ldots, n+1)$, $\overline{V}$ be the volume of the pedal simplex $\overline{\Omega}_n$, $V(i)$ and $\overline{V}(i)$ denote the volumes of two n-dimensional simplices $\Omega_n(i) = A_1 \cdots A_{i-1}PA_{i+1} \cdots A_{n+1}$ and $\overline{\Omega}_n(i) = B_1 \cdots B_{i-1}PB_{i+1} \cdots B_{n+1}$, respectively. Then we have an inequality for volumes of just defined n-simplices as follows.

Theorem 3.1. *Let P be an arbitrary point inside n-dimensional simplex Ω_n and λ_i $(i = 1, 2, \ldots, n+1)$ positive real numbers, then we have*

$$\sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1}\lambda_{i+1} \cdots \lambda_{n+1}\overline{V}(i) \leq \frac{1}{n^n}\left[\sum_{i=1}^{n} \lambda_i V(i)\right]^n V^{1-n}, \tag{3.1}$$

with equality if the simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$.

Now we state some applications of Theorem 3.1.

If taking $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ in inequality (3.1), we have

$$\sum_{i=1}^{n+1} \overline{V}(i) \leq \frac{1}{n^n}\left[\sum_{i=1}^{n+1} V(i)\right]^n \cdot V^{1-n}. \tag{3.2}$$

Since the point P is in the interior of the simplex Ω_n, then

$$\sum_{i=1}^{n+1} \overline{V}(i) = \overline{V}, \quad \sum_{i=1}^{n+1} V(i) = V. \tag{3.3}$$

Using (3.2) and (3.3) we obtain an inequality for the volume of the pedal simplex $\overline{\Omega}_n$ of the point P with respect to the simplex Ω_n as follows.

Corollary 3.2. *Let P be an arbitrary point inside the n-simplex Ω_n, then we have*

$$\overline{V} \leq \frac{1}{n^n} V, \tag{3.4}$$

with equality if simplex Ω_n is regular and P is the circumcenter of Ω_n.

Corollary 3.3. *Let P be an arbitrary point inside the n-simplex Ω_n, then we have*

$$\sum_{i=1}^{n+1} V(i) \cdot \overline{V}(i) \leq \frac{1}{(n+1)n^n} V^2, \tag{3.5}$$

with equality if the simplex Ω_n is regular and P is the circumcenter of Ω_n.

Proof. Let $\lambda_i = [V(i)]^{-1}$ $(i = 1, 2, \dots, n+1)$ in inequality (3.1); we get

$$\sum_{i=1}^{n+1} V(i) \cdot \overline{V}(i) \leq \left(\frac{n+1}{n}\right)^n V^{1-n} \prod_{j=1}^{n+1} V(j). \tag{3.6}$$

Using the arithmetic-geometric mean inequality and equality (3.3), we have

$$\sum_{i=1}^{n+1} V(i)\cdot \overline{V}(i) \le \left(\frac{n+1}{n}\right)^n V^{1-n}\left[\frac{1}{n+1}\sum_{j=1}^{n+1} V(j)\right]^{n+1} = \frac{1}{(n+1)n^n}V^2.$$

It is easy to see that equality in (3.5) holds if the simplex Ω_n is regular and the point P is the circumcenter of Ω_n. □

Proof of Theorem 3.1. Let h_i be the altitude of simplex Ω_n from vertex A_i, $\overrightarrow{PB_i} = r_i\boldsymbol{e_i}$, where $\boldsymbol{e_i}$ is the unit outer normal vector of the i-th side face $f_i = A_1\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$ of the simplex Ω_n, and ${}^n\sin\alpha_k$ be the n-dimensional sine of the k-th corner α_k of the simplex Ω_n. Wang and Yang [8] proved that

$$(3.7)\qquad {}^n\sin\alpha_n = [\det(e_i\cdot e_j)_{ij\neq k}]^{\frac{1}{2}} \qquad (k=1,2,\dots,n+1).$$

By the formula for the volume of an n-simplex and (3.7), we have

$$(3.8)\qquad \overline{V}(i) = \frac{1}{n!}[\det(r_l r_k \boldsymbol{e_l}\cdot\boldsymbol{e_k})_{l,k\neq i}]^{\frac{1}{2}} = \frac{1}{n!}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} r_j\right)\cdot {}^n\sin\alpha_i.$$

Using (3.8), (2.1) and $nV = h_iF_i$, we get that

$$\sum_{i=1}^{n+1}\lambda_1\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_{n+1}\overline{V}(i)$$

$$= \frac{1}{n!}\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1}\lambda_j r_j\right)\cdot {}^n\sin\alpha_i$$

$$= \frac{1}{n!}\sum_{i=1}^{n+1}\left\{\left(\prod_{\substack{j=1\\j\neq i}}^{n+1}\lambda_j r_j\right)(nV)^{n-1}\left[(n-1)!\cdot\prod_{\substack{j=1\\j\neq i}}^{n+1}F_j\right]^{-1}\right\}$$

$$= [(n!)^2\cdot V]^{-1}\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\j\neq i}}^{n+1}\lambda_j r_j h_j\right),$$

i.e.

$$(n!)^2 V\sum_{i=1}^{n+1}\lambda_1\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_{n+1}\overline{V}(i) = \sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\j\neq i}}^{n+1}\lambda_j r_j h_j\right). \tag{3.9}$$

Taking $s = n-1,\ l = n$ in inequality (2.14), we get

$$\left[\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\j\neq i}}^{n+1}x_j\right)F_i^2\right]^n \geq \frac{n^{3n}}{(n!)^2}\left(\sum_{i=1}^{n+1}x_i\right)\left(\prod_{i=1}^{n+1}x_i\right)^{n-1}V^{2(n-1)}. \tag{3.10}$$

Let $x_i = (\lambda_i r_i h_i)^{-1}$ $(i = 1, 2, \ldots, n+1)$ in inequality (3.10). Then we have

$$\left(\sum_{i=1}^{n+1}\lambda_i r_i h_i F_i^2\right)^n \geq \frac{n^{3n}}{(n!)^2}\left[\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\j\neq i}}^{n+1}\lambda_j r_j h_j\right)F_i^2\right]\cdot V^{2(n-1)}. \tag{3.11}$$

Using inequality (3.11) and $r_i F_i = nV(i),\ h_i F_i = nV$, we get

$$V^n \left[\sum_{i=1}^{n+1} \lambda_i V(i) \right]^n \geq \frac{n^n}{(n!)^2} V^{2(n-1)} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right). \tag{3.12}$$

Substituting equality (3.9) into inequality (3.12) we get inequality (3.1). It is easy to prove that equality in (3.1) holds if simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$. Theorem 3.1 is proved. □

Finally, we shall establish some inequalities for volumes of two n-simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P.Chiakui inequality will be given.

Let a_i $(i = 1, 2, 3)$ denote the sides of the triangle $A_1 A_2 A_3$ with area Δ, and a_i' $(i = 1, 2, 3)$ denote the sides of the triangle $A_1' A_2' A_3'$ with area Δ', then

$$\sum_{i=1}^{3} a_i^2 \left(\sum_{j=1}^{3} \left(a_j'\right)^2 - 2\left(a_i'\right)^2 \right) \geq 16\Delta\Delta', \tag{3.13}$$

with equality if and only if $\Delta A_1 A_2 A_3$ is similar to $\Delta A_1' A_2' A_3'$.

Inequality (3.13) is the well-known Neuberg-Pedoe inequality.

In 1984, P. Chiakui [9] proved the following sharpening of the Neuberg-

Pedoe inequality:

$$\sum_{i=1}^{3} a_i^2 \left(\sum_{j=1}^{3} (a_j')^2 - 2(a_i')^2 \right) \geq 8 \left(\frac{(a_1')^2 + (a_2')^2 + (a_3')^2}{a_1^2 + a_2^2 + a_3^2} \Delta^2 + \frac{a_1^2 + a_2^2 + a_3^2}{(a_1')^2 + (a_2')^2 + (a_3')^2} (\Delta')^2 \right), \tag{3.14}$$

with equality if and only if $\Delta A_1 A_2 A_3$ is similar to $\Delta A_1' A_2' A_3'$.

Recently, Leng Gangson [5] has extended inequality (3.14) to the edge lengths and volumes of two n-simplices. In this paper, we shall extend inequality (3.14) to the volumes of two n-simplices and the contents of their side faces. As a corollary, we get a generalization to several dimensions of the Neuberg-Pedoe inequality. Let A_i $(i = 1, 2, \ldots, n+1)$ be the vertices of n-simplex Ω_n in E^n, V the volume of the simplex Ω_n and $F_i (n-1)$- dimensional content of the $(n-1)$-dimensional face $f_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ of Ω_n. For two n-simplices Ω_n and Ω_n' and real numbers $\alpha, \beta \in (0, 1]$, we put

$$\sigma_n(\alpha) = \sum_{i=1}^{n+1} F_i^{\alpha}, \quad \sigma_n(\beta) = \sum_{i=1}^{n+1} (F_i')^{\beta}, \quad b_n = \frac{n^3}{n+1} \left(\frac{n+1}{n!^2} \right)^{\frac{1}{n}}. \tag{3.15}$$

We obtain an inequality for volumes of two n-simplices as follows.

Theorem 3.4. *For any two n-dimensional simplices Ω_n and Ω_n' and two arbi-*

trary real numbers $\alpha, \beta \in (0,1]$, *we have*

$$\sum_{i=1}^{n+1} F_i^{\alpha} \left(\sum_{j=1}^{n+1} \left(F_j'\right)^{\beta} - 2\left(F_i'\right)^{\beta} \right) \geq \frac{(n-1)^2}{2} \left[b_n^{\alpha} \frac{\sigma_n(\beta)}{\sigma_n(\alpha)} V^{2(n-1)\alpha/n} + b_n^{\beta} \frac{\sigma_n(\alpha)}{\sigma_n(\beta)} \left(V'\right)^{2(n-1)\beta/n} \right]. \tag{3.16}$$

Equality holds if and only if simplices Ω_n *and* Ω_n' *are regular.*

Using inequality (3.16) and the arithmetic-geometric mean inequality, we get the following corollary.

Corollary 3.5. *For any two* n*-dimensional simplices* Ω_n *and* Ω_n' *and two arbitrary real numbers* $\alpha, \beta \in (0,1]$, *we have*

$$\sum_{i=1}^{n+1} F_i^{\alpha} \left(\sum_{j=1}^{n+1} \left(F_j'\right)^{\beta} - 2\left(F_i'\right)^{\beta} \right) \geq b_n^{(\alpha+\beta)/2} (n^2-1) \left(V^{\alpha} \left(V'\right)^{\beta}\right)^{(n-1)/n}. \tag{3.17}$$

Equality holds if and only if simplices Ω_n *and* Ω_n' *are regular.*

If we let $\alpha = \beta$ in Corollary 3.5, we get Leng Gangson's inequality [6] as follows. For any $\theta \in (0,1]$ we have

$$\sum_{i=1}^{n+1} F_i^{\theta} \left(\sum_{j=1}^{n+1} \left(F_j'\right)^{\theta} - 2\left(F_i'\right)^{\theta} \right) \geq b_n^{\theta} (n^2-1) \left(VV'\right)^{(n-1)\theta/n}, \tag{3.18}$$

with equality if and only if simplices Ω_n and Ω_n' are regular.

To prove Theorem 3.4, we need some lemmas as follows.

Lemma 3.6. *For an n-simplex Ω_n and arbitrary number $\alpha \in (0,1]$, we have*

$$\frac{\prod_{i=1}^{n+1} F_i^{2\alpha}}{\sum_{i=1}^{n+1} F_i^{2\alpha}} \geq \frac{1}{(n+1)^{(n-1)\alpha+1}} \left[\frac{n^{3n}}{(n!)^2}\right]^{\alpha} V^{2(n-1)\alpha}, \tag{3.19}$$

with equality if and only if simplex Ω_n is regular.

Proof. If taking $l = n,\ s = n-1$ and $x_i = F_i^2$ $(i = 1, 2, \ldots, n+1)$ in inequality (2.14), we get an inequality as follows

$$\frac{(n+1)^n (n!)^2}{n^{3n}} \prod_{i=1}^{n+1} F_i^2 \geq V^{2(n-1)} \sum_{i=1}^{n+1} F_i^2,$$

or

$$\frac{(n+1)^{n\alpha} (n!)^{2\alpha}}{n^{3n\alpha}} \prod_{i=1}^{n+1} F_i^{2\alpha} \geq V^{2(n-1)\alpha} \left(\sum_{i=1}^{n+1} F_i^2\right)^{\alpha}. \tag{3.20}$$

It is easy to prove that equality in (3.20) holds if and only if simplex Ω_n is regular. From inequality (3.20) we know that inequality (3.19) holds for $\alpha = 1$. For $\alpha \in (0,1)$, using inequality (3.20) and the well-known inequality

$$\sum_{i=1}^{n+1} F_i^2 \geq (n+1) \left(\frac{1}{n+1} \sum_{i=1}^{n+1} F_i^{2\alpha}\right)^{\frac{1}{\alpha}}, \tag{3.21}$$

we get inequality (3.19). It is easy to see that equality in (3.19) holds if and only if the simplex Ω_n is regular. □

Lemma 3.7. *For an n-simplex $\Omega_n (n \geq 3)$ and an arbitrary number $\alpha \in (0,1]$, we have*

$$\left(\sum_{i=1}^{n+1} F_i^\alpha\right)^2 - 2\sum_{i=1}^{n+1} F_i^{2\alpha} \geq b_n^\alpha (n^2-1) V^{2(n-1)\alpha/n}, \tag{3.22}$$

with equality if and only if the simplex Ω_n is regular.

For the proof of Lemma 3.7, the reader is referred to [6].

Lemma 3.8. *Let a_i $(i = 1,2,3)$ and Δ denote the sides and the area of the triangle $(A_1A_2A_3)$, respectively. For arbitrary number $\alpha \in (0,1]$, denote by Δ_α the area of the triangle $(A_1A_2A_3)_\alpha$ with sides a_i^α $(i = 1,2,3)$, then the following inequality holds*

$$\Delta_\alpha^2 \geq \frac{3}{16}\left(\frac{16}{3}\Delta^2\right)^\alpha. \tag{3.23}$$

For $\alpha \neq 1$, equality holds if and only if $a_1 = a_2 = a_3$.

For the proof of Lemma 3.8, the reader is referred to [9].

Lemma 3.9. *Let numbers $x_i > 0, y_i > 0$ $(i = 1,2,\ldots,n+1)$, $\sigma_n = \sum_{i=1}^{n+1} x_i$, $\sigma'_n = \sum_{i=1}^{n+1} y_i$, then*

$$\begin{aligned} \sigma_n\sigma'_n - 2\sum_{i=1}^{n+1} x_i y_i \\ \geq \frac{1}{2}\left[\frac{\sigma'_n}{\sigma_n}\left(\sigma_n^2 - 2\sum_{i=1}^{n+1} x_i^2\right) + \frac{\sigma_n}{\sigma'_n}\left((\sigma'_n)^2 - 2\sum_{i=1}^{n+1} y_i^2\right)\right], \end{aligned} \tag{3.24}$$

with equality if and only if

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_{n+1}}{x_{n+1}}.$$

Proof. Inequality (3.24) is

$$\frac{\sigma_n'}{\sigma_n}\sum_{i=1}^{n+1} x_i^2 + \frac{\sigma_n}{\sigma_n'}\sum_{i=1}^{n+1} y_i^2 \geq 2\sum_{i=1}^{n+1} x_i y_i. \tag{3.25}$$

Now we prove that inequality (3.25) holds. Using the arithmetic-geometric mean inequality, we have

$$\frac{\sigma_n'}{\sigma_n}x_i^2 + \frac{\sigma_n}{\sigma_n'}y_i^2 \geq 2x_i y_i \qquad (i = 1, 2, \ldots, n+1).$$

Adding up those $n+1$ inequalities, we get inequality (3.25). Equality in (3.25) holds if and only if $\frac{\sigma_n'}{\sigma_n}x_i^2 = \frac{\sigma_n}{\sigma_n'}y_i^2$ $(i = 1, 2, \ldots, n+1)$, i.e.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_{n+1}}{x_{n+1}} = \frac{\sigma_n'}{\sigma_n}.$$

□

Proof of Theorem 3.4. For $n = 2$, consider two triangles $(A_1A_2A_3)_\alpha$ and $(A_1'A_2'A_3')_\beta$. Using inequality (3.14) and Lemma 3.8, we have

$$\sum_{i=1}^{3} a_i^\alpha \left(\sum_{j=1}^{3} \left(a_j'\right)^\beta - 2\left(a_i'\right)^\beta\right) \geq \frac{1}{2}\left[b_2^\alpha \frac{\sigma_2'(\beta)}{\sigma_2(\alpha)}\Delta^\alpha + b_2^\beta \frac{\sigma_2(\alpha)}{\sigma_2'(\beta)}\left(\Delta'\right)^\beta\right]. \tag{3.26}$$

Equality in (3.26) holds if and only if $a_1 = a_2 = a_3$ and $a_1' = a_2' = a_3'$. Hence, inequality (3.16) holds for $n = 2$.

For $n \geq 3$, taking $x_i = F_i^\alpha,\ y_i = (F_i')^\beta\ \ (i = 1, 2, \dots, n+1)$ in inequality (3.24), we get

$$
\begin{aligned}
\sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{j=1}^{n+1} (F_j')^\beta - 2 (F_i')^\beta \right)
&= \left(\sum_{i=1}^{n+1} F_i^\alpha \right) \left(\sum_{i=1}^{n+1} (F_i')^\beta \right) - 2 \sum_{i=1}^{n+1} F_i^\alpha (F_i')^\beta \\
&\geq \frac{1}{2} \left\{ \frac{\sigma_n'(\beta)}{\sigma_n(\alpha)} \left[\left(\sum_{i=1}^{n+1} F_i^\alpha \right)^2 - 2 \sum_{i=1}^{n+1} F_i^{2\alpha} \right] \right. \\
&\qquad \left. + \frac{\sigma_n(\alpha)}{\sigma_n'(\beta)} \left[\left(\sum_{i=1}^{n+1} (F_i')^\beta \right)^2 - 2 \sum_{i=1}^{n+1} F_i^{2\beta} \right] \right\}.
\end{aligned} \tag{3.27}
$$

Using inequality (3.27) and Lemma 3.7, we get

$$
\sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{i=1}^{n+1} (F_j')^\beta \right) \geq \frac{n^2 - 1}{2} \left[b_n^\alpha \frac{\sigma_n'(\beta)}{\sigma_n(\alpha)} V^{2(n-1)\alpha/n} + b_n^\beta \frac{\sigma_n(\alpha)}{\sigma_n'(\beta)} V^{2(n-1)\beta/n} \right].
$$

Hence, inequality (3.16) is true for $n \geq 3$. For $n \geq 3$, it is easy to see that equality in (3.16) holds if and only if two simplices Ω_n and σ_n' are regular. Theorem 3.4 is proved. □

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