# INEQUALITIES ON THE LAMBERT $W$ FUNCTION AND HYPERPOWER FUNCTION

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*Abstract:* In this note, we obtain inequalities for the Lambert $W$ function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. Also, we get upper and lower bounds for the hyperpower function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.



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## 1. Introduction

The Lambert $W$ function $W(x)$, is defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. For $-e^{-1} \leq x < 0$, there are two possible values of $W(x)$, which we take values not less than $-1$. The history of the function goes back to J. H. Lambert (1728-1777). One can find in [2] a more detailed definition of $W$ as a complex variable function, some historical background and various applications of it in Mathematics and Physics. The expansion

$$W(x) = \log x - \log\log x + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km}\frac{(\log\log x)^m}{(\log x)^{k+m}},$$

holds true for large values of $x$, with $c_{km} = \frac{(-1)^k}{m!}S[k+m, k+1]$, where $S[k+m, k+1]$ is Stirling cycle number [2]. The series in the above expansion is absolutely convergent and it can be rearranged into the form

$$W(x) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2-2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + O\left(\left(\frac{L_2}{L_1}\right)^4\right),$$

where $L_1 = \log x$ and $L_2 = \log\log x$. Note that by $\log$ we mean logarithm in the base $e$. Since the Lambert $W$ function appears in some problems in Mathematics, Physics and Engineering, it is very useful to have some explicit bounds for it. In [5] it is shown that

(1.1) $$\log x - \log\log x < W(x) < \log x,$$

where the left hand side holds true for $x > 41.19$ and the right hand side holds true for $x > e$. The aim of the present paper is to obtain some sharper bounds.



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## 2. Some Sharp Bounds for the Lambert $W$ Function

It is easy to see that $W(-e^{-1}) = -1$, $W(0) = 0$ and $W(e) = 1$. Also, for $x > 0$, since $W(x)e^{W(x)} = x > 0$ and $e^{W(x)} > 0$, we have $W(x) > 0$. An easy calculation yields that

$$\frac{d}{dx}W(x) = \frac{W(x)}{x(1+W(x))}.$$

Thus, $x\frac{d}{dx}W(x) > 0$ holds true for $x > 0$ and consequently $W(x)$ is strictly increasing for $x > 0$ (and also for $-e^{-1} \leq x \leq 0$, but not for this reason).

**Theorem 2.1.** *For every $x \geq e$, we have*

$$\log x - \log\log x \leq W(x) \leq \log x - \frac{1}{2}\log\log x, \tag{2.1}$$

*with equality holding only for $x = e$. The coefficients $-1$ and $-\frac{1}{2}$ of $\log\log x$ both are best possible for the range $x \geq e$.*

*Proof.* For the given constant $0 < p \leq 2$ consider the function

$$f(x) = \log x - \frac{1}{p}\log\log x - W(x),$$

for $x \geq e$. Obviously,

$$\frac{d}{dx}f(x) = \frac{p\log x - 1 - W(x)}{px(1+W(x))\log x},$$

and if $p = 2$, then

$$\frac{d}{dx}f(x) = \frac{(\log x - W(x)) + (\log x - 1)}{2x(1+W(x))\log x}.$$


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Considering the right hand side of (1.1), we have $\frac{d}{dx}f(x) > 0$ for $x > e$ and consequently $f(x) > f(e) = 0$, and this gives right hand side of (2.1). Trivially, equality only holds for $x = e$. If $0 < p < 2$, then $\frac{d}{dx}f(e) = \frac{p-2}{2ep} < 0$, and this yields that the coefficient $-\frac{1}{2}$ of $\log\log x$ in the right hand side of (2.1) is the best possible for the range $x \geq e$.

For the other side, note that $\log W(x) = \log x - W(x)$ and the inequality $\log W(x) \leq \log\log x$ holds for $x \geq e$, because of the right hand side of (1.1). Thus, $\log x - W(x) \leq \log\log x$ holds for $x \geq e$ with equality only for $x = e$. The sharpness of (2.1) with coefficient $-1$ for $\log\log x$ comes from the relation $\lim_{x\to\infty}(W(x) - \log x + \log\log x) = 0$. This completes the proof. $\square$

Now, we try to obtain some upper bounds for the function $W(x)$ with the main term $\log x - \log\log x$. To do this, we need the following lemma.

**Lemma 2.2.** *For every $t \in \mathbb{R}$ and $y > 0$, we have*

$$(t - \log y)e^t + y \geq e^t,$$

*with equality for $t = \log y$.*

*Proof.* Letting $f(t) = (t - \log y)e^t + y - e^t$, we have $\frac{d}{dt}f(t) = (t - \log y)e^t$ and $\frac{d^2}{dt^2}f(t) = (t + 1 - \log y)e^t$. Now, we observe that $f(\log y) = \frac{d}{dt}f(\log y) = 0$ and $\frac{d^2}{dt^2}f(\log y) = y > 0$. These show that the function $f(t)$ takes its only minimum value (equal to 0) at $t = \log y$, which yields the result of Lemma 2.2. $\square$

**Theorem 2.3.** *For $y > \frac{1}{e}$ and $x > -\frac{1}{e}$ we have*

$$W(x) \leq \log\left(\frac{x+y}{1+\log y}\right), \tag{2.2}$$

*with equality only for $x = y\log y$.*


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*Proof.* Using the result of Lemma 2.2 with $t = W(x)$, we get

$$(W(x) - \log y)e^{W(x)} - (e^{W(x)} - y) \geq 0,$$

which, considering $W(x)e^{W(x)} = x$, gives $(1 + \log y)e^{W(x)} \leq x + y$ and this is desired inequality for $y > \frac{1}{e}$ and $x > -\frac{1}{e}$. The equality holds when $W(x) = \log y$, i.e., $x = y \log y$. $\square$

**Corollary 2.4.** *For $x \geq e$ we have*

$$\log x - \log\log x \leq W(x) \leq \log x - \log\log x + \log(1 + e^{-1}), \tag{2.3}$$

*where equality holds in the left hand side for $x = e$ and in the right hand side for $x = e^{e+1}$.*

*Proof.* Consider (2.2) with $y = \frac{x}{e}$, and the left hand side of (2.1). $\square$

*Remark* 1. Taking $y = x$ in (2.2) we get $W(x) \leq \log x - \log\left(\frac{1+\log x}{2}\right)$, which is sharper than the right hand side of (2.1).

**Theorem 2.5.** *For $x > 1$, we have*

$$W(x) \geq \frac{\log x}{1 + \log x}(\log x - \log\log x + 1), \tag{2.4}$$

*with equality only for $x = e$.*

*Proof.* For $t > 0$ and $x > 1$, let

$$f(t) = \frac{t - \log x}{\log x} - (\log t - \log\log x).$$


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We have $\frac{d}{dt}f(t) = \frac{1}{\log x} - \frac{1}{t}$ and $\frac{d^2}{dt^2}f(t) = \frac{1}{t^2} > 0$. Now, we observe that $\frac{d}{dt}f(\log x) = 0$ and so

$$\min_{t>0} f(t) = f(\log x) = 0.$$

Thus, for $t > 0$ and $x > 1$ we have $f(t) \geq 0$, with equality at $t = \log x$. Putting $t = W(x)$ and simplifying, we get the result, with equality at $W(x) = \log x$, or, equivalently, at $x = e$. $\square$

**Corollary 2.6.** *For $x > 1$ we have*

$$W(x) \leq (\log x)^{\frac{\log x}{1+\log x}}.$$

*Proof.* This refinement of the right hand side of (1.1) can be obtained by simplifying (2.4) with $W(x) = \log x - \log W(x)$. $\square$

The bounds we have obtained up to now have the form $W(x) = \log x - \log\log x + O(1)$. Now, we give bounds with the error term $O(\frac{\log\log x}{\log x})$ instead of $O(1)$.

**Theorem 2.7.** *For every $x \geq e$ we have*

$$\log x - \log\log x + \frac{1}{2}\frac{\log\log x}{\log x} \leq W(x) \leq \log x - \log\log x + \frac{e}{e-1}\frac{\log\log x}{\log x}, \tag{2.5}$$

*with equality only for $x = e$.*

*Proof.* Taking the logarithm of the right hand side of (2.1), we have

$$\log W(x) \leq \log\left(\log x - \frac{1}{2}\log\log x\right) = \log\log x + \log\left(1 - \frac{\log\log x}{2\log x}\right).$$

Using $\log W(x) = \log x - W(x)$, we get

$$W(x) \geq \log x - \log\log x - \log\left(1 - \frac{\log\log x}{2\log x}\right),$$


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which, considering $-\log(1-t) \geq t$ for $0 \leq t < 1$ (see [1]) with $t = \frac{\log\log x}{2\log x}$, implies the left hand side of (2.5). To prove the other side, we take the logarithm of the left hand side of (2.1) to get

$$\log W(x) \geq \log(\log x - \log\log x) = \log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Again, using $\log W(x) = \log x - W(x)$, we obtain

$$W(x) \leq \log x - \log\log x - \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Now we use the inequality $-\log(1-t) \leq \frac{t}{1-t}$ for $0 \leq t < 1$ (see [1]) with $t = \frac{\log\log x}{\log x}$, to get

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \leq \frac{\log\log x}{\log x}\left(1 - \frac{\log\log x}{\log x}\right)^{-1} \leq \frac{1}{m}\,\frac{\log\log x}{\log x},$$

where

$$m = \min_{x \geq e}\left(1 - \frac{\log\log x}{\log x}\right) = 1 - \frac{1}{e}.$$

Thus, we have

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \leq \frac{e}{e-1}\,\frac{\log\log x}{\log x},$$

which gives the desired bounds. This completes the proof. $\square$



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## 3. Studying the Hyperpower Function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$

Consider the hyperpower function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. One can define this function as the limit of the sequence $\{h_n(x)\}_{n\in\mathbb{N}}$ with $h_1(x) = x$ and $h_{n+1}(x) = x^{h_n(x)}$. It is proven that this sequence converges if and only if $e^{-e} \leq x \leq e^{\frac{1}{e}}$ (see [4] and references therein). This function satisfies the relation $h(x) = x^{h(x)}$, which, on taking the logarithm of both sides and a simple calculation yields

$$h(x) = \frac{W(\log(x^{-1}))}{\log(x^{-1})}.$$

In this section we obtain some explicit upper and lower bounds for this function. Since the obtained bounds for $W(x)$ hold for large values of $x$ and since for such values of $x$ the value of $\log(x^{-1})$ is negative, we cannot use these bounds to approximate $h(x)$.

**Theorem 3.1.** *Taking* $\lambda = e - 1 - \log(e-1) = 1.176956974\ldots$, *for* $e^{-e} \leq x \leq e^{\frac{1}{e}}$ *we have*

$$\frac{1+\log(1-\log x)}{1-2\log x} \leq h(x) \leq \frac{\lambda+\log(1-\log x)}{1-2\log x}, \tag{3.1}$$

*where equality holds in the left hand side for* $x = 1$ *and in the right hand side for* $x = e^{\frac{1}{e}}$.

*Proof.* For $t > 0$, we have $t \geq \log t + 1$, which taking $t = z - \log z$ with $z > 0$, implies

$$z\left(1 - 2\log(z^{\frac{1}{z}})\right) \geq \log\left(1 - \log(z^{\frac{1}{z}})\right) + 1,$$


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and putting $z^{\frac{1}{z}} = x$, or equivalently $z = h(x)$, yields that $h(x)(1 - 2\log x) \geq \log(1 - \log x) + 1$; this is the left hand side (3.1), since $1 - 2\log x$ is positive for $e^{-e} \leq x \leq e^{\frac{1}{e}}$. Note that equality holds for $t = z = x = 1$.

For the right hand side, we define $f(z) = z - \log z$ with $\frac{1}{e} \leq z \leq e$. We immediately see that $1 \leq f(z) \leq e - 1$; in fact it takes its minimum value 1 at $z = 1$. Also, consider the function $g(t) = \log t - t + \lambda$ for $1 \leq t \leq e - 1$, with $\lambda = e - 1 - \log(e - 1)$. Since $\frac{d}{dt}g(t) = \frac{1}{t} - 1$ and $g(e - 1) = 0$, we obtain the inequality $\log t - t + \lambda \geq 0$ for $1 \leq t \leq e - 1$, and putting $t = z - \log z$ with $\frac{1}{e} \leq z \leq e$ in this inequality, we obtain

$$\log(1 - \log z) + \lambda \geq z\left(1 - 2\log(z^{\frac{1}{z}})\right).$$

Taking $z^{\frac{1}{z}} = x$, or equivalently $z = h(x)$ yields the right hand side (3.1). Note that equality holds for $x = e^{\frac{1}{e}}$ $(z = e, t = e - 1)$. This completes the proof. $\square$


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