A GEOMETRIC INEQUALITY OF THE GENERALIZED ERDÖS-MORDELL TYPE

YU-DONG WU, CHUN-LEI YU

Department of Mathematics
Zhejiang Xinchang High School
Shaoxing 312500, Zhejiang
People's Republic of China
EMail: yudong.wu@yahoo.com.cn
seetill@126.com

ZHI-HUA ZHANG

Department of Mathematics
Shili Senior High School in Zixing
Chenzhou 423400, Hunan
People's Republic of China
EMail: zxzh1234@163.com

Received: 20 April, 2009

Accepted: 09 October, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: Primary 51M16.

Key words: Geometric inequality, triangle, Erdös-Mordell inequality, Hayashi's inequality, Klamkin's inequality.

Abstract: In this short note, we solve an interesting geometric inequality problem relating to two points in triangle posed by Liu [7], and also give two corollaries.

Acknowledgements: Dedicated to Mr. Ting-Feng Dong on the occasion of his 55th birthday.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction and Main Results

Let P, Q be two arbitrary interior points in $\triangle ABC$, and let a, b, c be the lengths of its sides, S the area, R the circumradius and r the inradius, respectively. Denote by R_1, R_2, R_3 and r_1, r_2, r_3 the distances from P to the vertices A, B, C and the sides BC, CA, AB, respectively. For the interior point Q, define D_1, D_2, D_3 and d_1, d_2, d_3 similarly (see Figure 1).

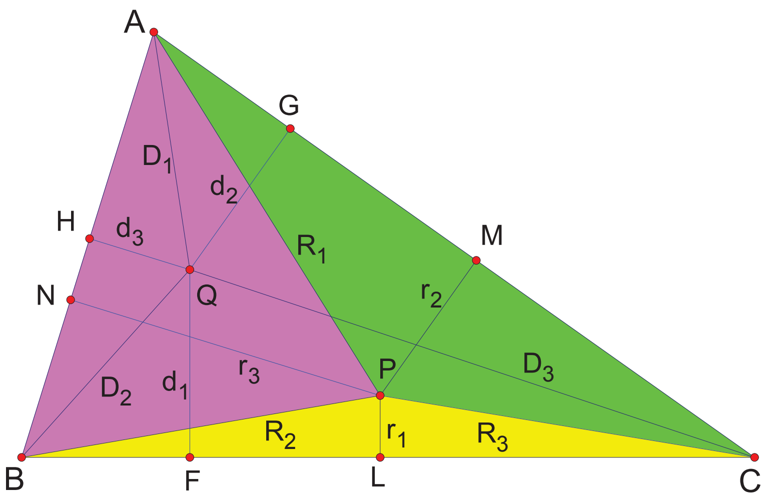

Figure 1:

The following well-known and elegant result (see [1, Theorem 12.13, pp.105])

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{1.1}$$

Erdös-Mordell Type Geometric Inequality
Yu-Dong Wu, Chun-Lei Yu and Zhi-Hua Zhang
vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

concerning R_i and r_i $(i = 1, 2, 3)$ is called the **Erdös-Mordell inequality**. Inequality (1.1) was generalized as follows [9, Theorem 15, pp. 318]:

$$R_1x^2 + R_2y^2 + R_3z^2 \geq 2(r_1yz + r_2zx + r_3xy) \tag{1.2}$$

for all $x, y, z \geq 0$.

And the special case $n = 2$ of [9, Theorem 8, pp. 315-316] states that

$$\sqrt{R_1D_1} + \sqrt{R_2D_2} + \sqrt{R_3D_3} \geq 2\left(\sqrt{r_1d_1} + \sqrt{r_2d_2} + \sqrt{r_3d_3}\right), \tag{1.3}$$

which also extends (1.1).

Recently, for all $x, y, z \geq 0$, J. Liu [8, Proposition 2] obtained

$$\begin{aligned} \sqrt{R_1D_1}x^2 + \sqrt{R_2D_2}y^2 + \sqrt{R_3D_3}z^2 \\ \geq 2\left(\sqrt{r_1d_1}yz + \sqrt{r_2d_2}zx + \sqrt{r_3d_3}xy\right) \end{aligned} \tag{1.4}$$

which generalizes inequality (1.3).

In 2008, J. Liu [7] posed the following interesting geometric inequality problem.

Problem 1. *For a triangle ABC and two arbitrary interior points P, Q, prove or disprove that*

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4(r_2r_3 + r_3r_1 + r_1r_2). \tag{1.5}$$

We will solve Problem 1 in this paper.

From inequality (1.5), we get

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4(d_2d_3 + d_3d_1 + d_1d_2).$$

Hence, we obtain the following result.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Corollary 1.1. *For any* $\triangle ABC$ *and two interior points* P, Q, *we have*

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4\sqrt{(r_2r_3 + r_3r_1 + r_1r_2)(d_2d_3 + d_3d_1 + d_1d_2)}. \tag{1.6}$$

From inequality (1.5), and by making use of an inversion transformation [2, pp.48-49] (see also [3, pp.108-109]) in the triangle, we easily get the following result.

Corollary 1.2. *For any* $\triangle ABC$ *and two interior points* P, Q, *we have*

$$\frac{D_1}{R_1r_1} + \frac{D_2}{R_2r_2} + \frac{D_3}{R_3r_3} \geq 4 \cdot |PQ| \cdot \left(\frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1}\right). \tag{1.7}$$

Remark 1. With one of Liu's theorems [8, Theorem 3], inequality (1.2) implies (1.4). However, we cannot determine whether inequalities (1.1) and (1.3) imply inequality (1.5) or inequality (1.6), or inequalities (1.5) and (1.3) imply inequality (1.1).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Preliminary Results

Lemma 2.1. *We have for any $\triangle ABC$ and an arbitrary interior point P that*

$$aR_1 \geq br_2 + cr_3, \tag{2.1}$$

$$bR_2 \geq cr_3 + ar_1, \tag{2.2}$$

$$cR_3 \geq ar_1 + br_2. \tag{2.3}$$

Proof. Inequalities (2.1) – (2.3) directly follow from the obvious fact

$$ar_1 + br_2 + cr_3 = 2S,$$

the formulas of the altitude

$$h_a = \frac{2S}{a}, \qquad h_b = \frac{2S}{b}, \qquad h_c = \frac{2S}{c},$$

and the evident inequalities [11]

$$R_1 + r_1 \geq h_a,$$

$$R_2 + r_2 \geq h_b,$$

$$R_3 + r_3 \geq h_c.$$

□

vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Lemma 2.2 ([4, 5]). *For real numbers* $x_1, x_2, x_3, y_1, y_2, y_3$ *such that*

$$x_1x_2 + x_2x_3 + x_3x_1 \geq 0$$

and

$$y_1y_2 + y_2y_3 + y_3y_1 \geq 0,$$

the inequality

$$(2.4)\quad (y_2 + y_3)x_1 + (y_3 + y_1)x_2 + (y_1 + y_2)x_3 \geq 2\sqrt{(x_1x_2 + x_2x_3 + x_3x_1)(y_1y_2 + y_2y_3 + y_3y_1)}$$

holds, with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}$.

Lemma 2.3 (Hayashi's inequality, [9, pp.297, 311]). *For any* $\triangle ABC$ *and an arbitrary point* P, *we have*

$$(2.5)\quad \frac{R_1R_2}{ab} + \frac{R_2R_3}{bc} + \frac{R_3R_1}{ca} \geq 1.$$

Equality holds if and only if P *is the orthocenter of the acute triangle* ABC *or one of the vertexes of triangle* ABC.

Lemma 2.4 (Klamkin's inequality, [6, 10]). *Let* A, B, C *be the angles of* $\triangle ABC$. *For positive real numbers* u, v, w, *the inequality*

$$(2.6)\quad u\sin A + v\sin B + w\sin C \leq \frac{1}{2}(uv + vw + wu)\sqrt{\frac{u + v + w}{uvw}}$$

holds, with equality if and only if $u = v = w$ *and* $\triangle ABC$ *is equilateral.*

Lemma 2.5. *For any* $\triangle ABC$ *and an arbitrary interior point* P, *we have*

$$(2.7)\quad \sqrt{abr_1r_2 + bcr_2r_3 + car_3r_1} \geq 2(r_2r_3 + r_3r_1 + r_1r_2).$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. Suppose that the actual barycentric coordinates of P are (x, y, z), Then $x =$ area of $\triangle PBC$, and therefore

$$\frac{x}{x+y+z} = \frac{\text{area}(\triangle PBC)}{S} = \frac{r_1 a}{bc \sin A} = \frac{2r_1}{bc} \cdot \frac{a}{2\sin A} = \frac{2Rr_1}{bc}.$$

Therefore

$$r_1 = \frac{bc}{2R} \cdot \frac{x}{x+y+z},$$
$$r_2 = \frac{ca}{2R} \cdot \frac{y}{x+y+z},$$
$$r_3 = \frac{ab}{2R} \cdot \frac{z}{x+y+z}.$$

Thus, inequality (2.7) is equivalent to

(2.8)
$$\frac{abc}{2R(x+y+z)}\sqrt{xy+yz+zx} \geq \frac{abc}{R(x+y+z)^2}\left(\frac{a}{2R}yz + \frac{b}{2R}zx + \frac{c}{2R}xy\right)$$

or

(2.9)
$$\frac{1}{2}(x+y+z)\sqrt{xy+yz+zx} \geq yz\sin A + zx\sin B + xy\sin C.$$

Inequality (2.9) follows from Lemma 2.4 by taking

$$(u, v, w) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right).$$

This completes the proof of Lemma 2.5. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Solution of Problem 1

Proof. In view of Lemmas 2.1 – 2.3 and 2.5, we have that

$$\begin{aligned}
&R_1D_1 + R_2D_2 + R_3D_3 \\
&\quad = aR_1 \cdot \frac{D_1}{a} + bR_2 \cdot \frac{D_2}{b} + cR_3 \cdot \frac{D_3}{c} \\
&\quad \geq (br_2 + cr_3) \cdot \frac{D_1}{a} + (cr_3 + ar_1) \cdot \frac{D_2}{b} + (ar_1 + br_2) \cdot \frac{D_3}{c} \\
&\quad \geq 2\sqrt{(abr_1r_2 + bcr_2r_3 + car_3r_1)\left(\frac{D_1D_2}{ab} + \frac{D_2D_3}{bc} + \frac{D_3D_1}{ca}\right)} \\
&\quad \geq 2\sqrt{abr_1r_2 + bcr_2r_3 + car_3r_1} \\
&\quad \geq 4(r_2r_3 + r_3r_1 + r_1r_2).
\end{aligned}$$

The proof of inequality (1.5) is thus completed. □

Erdös-Mordell Type Geometric Inequality
Yu-Dong Wu, Chun-Lei Yu and Zhi-Hua Zhang
vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] O. BOTTEMA, R.Ž. DJORDEVIĆ, R.R. JANIĆ, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[2] H.S.M. COXETER AND S.L. GREITZER, *Geometry Revisited*, Random House, New York, 1967.

[3] H.S.M. COXETER, *Introduction to Geometry*, 2nd ed., John Wiley & Sons, New York, London, Sydney, Toronto, 1969.

[4] PHAM HUU DUC, An unexpectedly useful inequality, *Mathematical Reflections*, **3**(1) (2008). [ONLINE: http://reflections.awesomemath.org/2008_1/unexpected_ineq.pdf].

[5] TRAN QUANG HUNG, On some geometric inequalities, *Mathematical Reflections*, **3**(3) (2008). [ONLINE: http://reflections.awesomemath.org/2008_3/on_some_geo_ineq.pdf].

[6] M.S. KLAMKIN, On a triangle inequality, *Crux Mathematicorum*, **10**(5) (1984), 139–140.

[7] J. LIU, Nine sine inequality, manuscript, 2008, 66. (in Chinese)

[8] J. LIU, The composite theorem of ternary quadratic inequalities and its applications, *RGMIA Res. Rep. Coll.*, **11**(4) (2008), Art. 13. [ONLINE http://eureka.vu.edu.au/~rgmia/v11n4/CTTQIApp.pdf].

[9] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Acad. Publ., Dordrecht, Boston, London, 1989.

vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756

[10] W.-X. SHEN, *Introduction to Simplices*, Hunan Normal University Press, Changsha, 2000, 179. (in Chinese)

[11] G. STEENSHOLT, Note on an elementary property of triangles, *Amer. Math. Monthly*, **63** (1956), 571–572.

vol. 10, iss. 4, art. 106, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756