Journal of Inequalities in Pure and Applied Mathematics

ON HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETERS

LEPING HE, MINGZHE GAO AND WEIJIAN JIA

Department of Mathematics and Computer Science,
Normal College, Jishou University,
Jishou Hunan, 416000
People's Republic of China.
EMail: lianheping@163.com

EMail: mingzhegao@163.com

EMail: jwj1959@163.com

ISSN (electronic): 1443-5756
108-03

volume 4, issue 5, article 94,
2003.

*Received 01 August, 2003;
accepted 28 August, 2003.*

Communicated by: L. Debnath

Abstract

In this paper, by means of a sharpening of Hölder's inequality, Hardy-Hilbert's integral inequality with parameters is improved. Some new inequalities are established.

2000 Mathematics Subject Classification: 26D15, 46C99
Key words: Hardy-Hilbert integral inequality, Hölder's inequality, Weight function, Beta function.

The authors thank the referee for his help and patience in improving the paper.

Contents

1. Introduction

Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f, g > 0$. If $0 < \int_0^\infty f^p(t)dt < +\infty$, $0 < \int_0^\infty g^q(t)dt < +\infty$, then

$$(1.1) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)\,dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)\,dt \right)^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is best possible. The inequality (1.1) is well known as Hardy-Hilbert's integral inequality. In recent years, some improvements and extensions of Hilbert's inequality and Hardy-Hilbert's inequality have been given in [2] – [6], Yang [2] gave a generalization of (1.1) as follows:

If $\lambda > 2 - \min\{p, q\}$, $\alpha < T \leqslant \infty$ then

$$(1.2) \qquad \int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy < \left\{ \int_\alpha^T \left[k_\lambda(p) - \theta_\lambda(p) \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{p+\lambda-2}{p}} \right] (t-\alpha)^{1-\lambda} f^p(t)dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^T \left[k_\lambda(p) - \theta_\lambda(q) \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{q+\lambda-2}{q}} \right] (t-\alpha)^{1-\lambda} g^q(t)dt \right\}^{\frac{1}{q}} \quad (T < \infty)$$

and

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy < k_\lambda(p)\left\{\int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t)dt\right\}^{\frac{1}{p}} \left\{\int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t)dt\right\}^{\frac{1}{q}}, \tag{1.3}$$

where

$$k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right),$$

$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du \qquad (r = p, q).$$

The main purpose of this paper is to build a few new inequalities which include improvements of the inequalities (1.2) and (1.3), and extensions of corresponding results in [3] – [5].

2. Lemmas and their Proofs

For convenience, we firstly introduce some notations:

$$(f^r, g^s) = \int_\alpha^T f^r(x) g^s(x)\,dx, \quad \|f\|_p = \left(\int_\alpha^T f^p(x)\,dx\right)^{\frac{1}{p}}, \quad \|f\|_2 = \|f\|.$$

We next introduce a function defined by

$$S_r(H,x) = \left(H^{r/2}, x\right) \|H\|_r^{-r/2},$$

where x is a parametric variable vector which is a variable unit vector. Under the general case, it is properly chosen such that the specific problems discussed are simplified.

Clearly, $S_r(H,x) = 0$ when the vector x selected is orthogonal to $H^{p/2}$. Throughout this paper, the exponent m indicates $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $\alpha < T \leqslant \infty$.

In order to verify our assertions, we need to build the following lemmas.

Lemma 2.1. *Let* $f(x), g(x) > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \|f\|_p < +\infty$ *and* $0 < \|g\|_q < +\infty$, *then*

$$(f,g) < \|f\|_p \|g\|_q (1-R)^m, \tag{2.1}$$

where $R = (S_p(f,h) - S_q(g,h))^2$, $\|h\| = 1$, $f^{p/2}(x)$, $g^{q/2}(x)$ *and* $h(x)$ *are linearly independent.*

Proof. First of all, we discuss the case of $p \neq q$. Without loss of generality, suppose that $p > q > 1$, since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$. Then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain,

$$
\begin{aligned}
(f,g) &= \int_a^T f(x)g(x)\,dx \\
&= \int_a^T \left(f \cdot g^{q/p}\right) g^{1-(q/p)} dx \\
&\leqslant \left(\int_a^T \left(f \cdot g^{q/p}\right)^R dx\right)^{\frac{1}{R}} \left(\int_a^T \left(g^{1-(q/p)}\right)^Q dx\right)^{\frac{1}{Q}} \\
&= \left(f^{p/2}, g^{q/2}\right)^{\frac{2}{p}} \|g\|_q^{q\left(1-\frac{2}{p}\right)}.
\end{aligned} \tag{2.2}
$$

And the equality in (2.2) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, the equality in (2.2) holds if and only if, there exists a c_1 such that $\left(f \cdot g^{q/p}\right)^R = c_1 \left(g^{1-(q/p)}\right)^Q$. It is easy to deduce that $f^{p/2} = c_1 g^{q/2}$.

In our previous paper [3], with the help of the positive definiteness of the Gram matrix, we established an important inequality of the form

$$
(\alpha,\beta)^2 \leqslant \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\|\, x - \|\beta\|\, y)^2 = \|\alpha\|^2 \|\beta\|^2 (1 - \overline{\gamma}) \tag{2.3}
$$

where $\overline{\gamma} = \left(\frac{y}{\|\alpha\|} - \frac{x}{\|\beta\|}\right)^2$, $x = (\beta,\gamma)$, $y = (\alpha,\gamma)$ with $\|\gamma\| = 1$ and $xy \geqslant 0$. The equality in (2.3) holds if and only if α and β are linearly dependent; or the vector γ is a linear combination of α and β, and $xy = 0$ but $x \neq y$. If α, β and γ in (2.3) are replaced by $f^{p/2}$, $g^{q/2}$ and h respectively, then we get

$$
\left(f^{p/2}, g^{q/2}\right)^2 \leqslant \|f\|_p^p \|g\|_q^q (1-R), \tag{2.4}
$$

where $R = (S_p(f,h) - S_q(g,h))^2$ with $||h|| = 1$. The equality in (2.4) holds if and only if $f^{p/2}$and $g^{q/2}$ are linearly dependent, or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) = 0$, but $\left(f^{p/2}, h\right) \neq \left(g^{q/2}, h\right)$. Since $f^{p/2}$ and $g^{q/2}$ are linearly independent, it is impossible to have equality in (2.4). Substituting (2.4) into (2.2), we obtain after simplifications

$$(f,g) < \|f\|_p \|g\|_q (1-R)^{\frac{1}{p}} . \tag{2.5}$$

Provided that $h(x)$ is properly chosen, then $R \neq 0$ is achieved. (The choice of $h(x)$ is quite flexible, as long as condition $\|h\| = 1$ is satisfied, on which we can refer to [3, 4], etc.). Noticing the symmetry of p and q, the inequality (2.1) follows from (2.5).

Next, we discuss the case of $p = q$. According to the hypothesis: when f, g and h are linearly independent, we immediately obtain from (2.3) the following result:

$$(f,g) < \|f\| \|g\| (1-\bar{r})^{\frac{1}{2}} ,$$

where $\bar{r} = \left(\frac{(f,h)}{\|f\|} - \frac{(g,h)}{\|g\|}\right)^2$, and $\|h\| = 1$. Thus the lemma is proved. □

Lemma 2.2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}, \alpha < T < \infty$. *Define the weight function* ω_λ *as follows:*

$$\omega_\lambda(\alpha, T, r, x) = \int_\alpha^T \frac{1}{(x+y-2\alpha)^\lambda} \left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{r}} dy \quad x \in (\alpha, T]. \tag{2.6}$$

Setting $\omega_\lambda(\alpha, \infty, r, x) = \lim_{T\to\infty} \omega_\lambda(\alpha, T, r, x)$ *and* $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$,

$$\bar{\theta}_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)(1-1/r)} du, \qquad (r = p, q), \tag{2.7}$$

then we have

$$\omega_\lambda(\alpha, \infty, r, x) = k_\lambda(p)(x-\alpha)^{1-\lambda}, \quad x \in (\alpha, \infty) \tag{2.8}$$

and

$$\begin{aligned} \omega_\lambda(\alpha, T, r, x) \\ < \left(k_\lambda(p) - \bar{\theta}(r)\left(\frac{x-\alpha}{T-\alpha}\right)^{1+(\lambda-2)(1-1/r)}\right)(x-\alpha)^{1-\lambda}, \quad x \in (\alpha, T), \end{aligned} \tag{2.9}$$

where $B(m, n)$ is the beta function.

The proof of this lemma is given in the paper [2]; it is omitted here.

3. Main Results

In order to state it conveniently, we need again to define the functions and introduce some notations

$$F = \frac{f(x)}{(x+y-2\alpha)^{\lambda/p}} \left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}}, \quad G = \frac{g(y)}{(x+y-2\alpha)^{\lambda/q}} \left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{pq}},$$

$$S_p(F, h_T) = \left\{\int_\alpha^T \int_\alpha^T F^{p/2} h_T dx dy\right\} \left\{\int_\alpha^T \int_\alpha^T F^p dx dy\right\}^{-\frac{1}{2}},$$

$$S_q(G, h_T) = \left\{\int_\alpha^T \int_\alpha^T G^{q/2} h_T dx dy\right\} \left\{\int_\alpha^T \int_\alpha^T G^q dx dy\right\}^{-\frac{1}{2}},$$

where $h_T = h_T(x,y)$ is a unit vector with two variants, namely

$$||h_T|| = \left\{\int_\alpha^T \int_\alpha^T h_T^2 dx dy\right\}^{\frac{1}{2}} = 1, \quad \alpha < T \leqslant \infty,$$

and $F^{p/2}$, $G^{q/2}$, h_T are linearly independent.

Theorem 3.1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, $\alpha < T \leqslant \infty$, $f(t), g(t) > 0$. *If*

$$0 < \int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t)\, dt < +\infty \quad \textit{and}$$

$$0 < \int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t)\, dt < +\infty,$$

then

(i) For $T < \infty$, we have

$$\int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy \tag{3.1}$$
$$< \left\{ \int_\alpha^T \left(k_\lambda(p) - \theta_\lambda(p) \left(\frac{t-\alpha}{T-\alpha} \right)^{(p+\lambda-2)/p} \right) (t-\alpha)^{1-\lambda} f^p(t)dt \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_\alpha^T \left(k_\lambda(p) - \theta_\lambda(q) \left(\frac{t-\alpha}{T-\alpha} \right)^{(q+\lambda+2)/q} \right) (t-\alpha)^{1-\lambda} g^q(t)dt \right\}^{\frac{1}{q}} (1-R_T)^m,$$

where

$$k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right),$$
$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda} \left(\frac{1}{u} \right)^{\frac{2-\lambda}{r}} du \quad (r = p, q).$$

(ii) For $T = \infty$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy \tag{3.2}$$
$$< k_\lambda(p) \left(\int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t)dt \right)^{\frac{1}{p}}$$
$$\times \left(\int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t)dt \right)^{\frac{1}{q}} (1-R_\infty)^m,$$

where $R_T = (S_p(F, h_T) - S_q(G, h_T))^2$,

$$h_T(x,y) = \begin{cases} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{a-x}}{(x+y-2a)^{\frac{1}{2}}} \left(\frac{x-a}{y-a}\right)^{\frac{1}{4}}, & T = \infty; \\ \frac{T-\alpha}{(x-\alpha)(y-\alpha)} e^{\left(1 - \frac{T-\alpha}{2(x-\alpha)} - \frac{T-\alpha}{2(y-\alpha)}\right)}, & T < \infty. \end{cases} \tag{3.3}$$

Proof. By Lemma 2.1, we get

$$\begin{aligned} \int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy &= \int_\alpha^T \int_\alpha^T FGdxdy \\ &\leqslant \left\{\int_\alpha^T \int_\alpha^T F^p dxdy\right\}^{\frac{1}{p}} \left\{\int_\alpha^T \int_\alpha^T G^q dxdy\right\}^{\frac{1}{q}} (1-R_T)^m \\ &= \left(\int_\alpha^T \omega_\lambda(\alpha,\beta,q,t) f^p(t)\,dt\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_\alpha^T \omega_\lambda(\alpha,\beta,p,t) g^q(t)\,dt\right)^{\frac{1}{q}} (1-R_T)^m, \end{aligned} \tag{3.4}$$

where $\omega_\lambda(\alpha, T, r, t) \quad (r = p, q)$ is the function defined by (2.6) .

Now notice that $\theta_\lambda(p) = \overline{\theta}_\lambda(q)$, $\theta_\lambda(q) = \overline{\theta}_\lambda(p)$ and substituting (2.9) and (2.8) into (3.4) respectively, the inequalities (3.1) and (3.2) follow.

It remains to discuss the expression of R_T. We may choose the function h_T indicated by (3.3).

When $T = \infty$, setting $s = x - \alpha$, $t = y - \alpha$, then

$$\|h_\infty\| = \left(\int_\alpha^\infty \int_\alpha^\infty h_\infty^2(x,y)dxdy\right)^{\frac{1}{2}}$$
$$= \left\{\frac{2}{\pi}\int_0^\infty e^{-2s}ds \int_0^\infty \frac{1}{s+t}\left(\frac{s}{t}\right)^{\frac{1}{2}} dt\right\}^{\frac{1}{2}} = 1.$$

When $T < \infty$, setting $\xi = \frac{T-\alpha}{x-\alpha}$, $\eta = \frac{T-\alpha}{y-\alpha}$, then we have

$$\|h_T\| = \left(\int_\alpha^T \int_\alpha^T h_T^2 dxdy\right)^{\frac{1}{2}}$$
$$= \left\{\int_\alpha^T \frac{T-\alpha}{(x-\alpha)^2} e^{\left(1-\frac{T-\alpha}{x-\alpha}\right)} dx \cdot \int_\alpha^T \frac{T-\alpha}{(y-\alpha)^2} e^{\left(1-\frac{T-\alpha}{y-\alpha}\right)} dy\right\}^{\frac{1}{2}}$$
$$= \left\{\int_1^\infty e^{1-\xi} d\xi \cdot \int_1^\infty e^{1-\eta} d\eta\right\}^{\frac{1}{2}} = 1.$$

According to Lemma 2.1 and the given h_T, we have $R_T = (S_p(F, h_T) - S_q(G, h_T))^2$. It is obvious that $F^{p/2}$, $G^{q/2}$ and h_T are linearly independent, so it is impossible for equality to hold in (3.4). Thus the proof of theorem is completed. □

Remark 3.1. *Clearly, the inequalities (3.1) and (3.2) are the improvements of (1.2) and (1.3) respectively .*

Owing to $p, q > 1$, when $\lambda = 1, 2, 3$, the condition $\lambda > 2 - \min(p, q)$ is satisfied, then we have

$$\theta_1(r) = \int_0^1 \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{\gamma}} du > \int_0^1 \frac{1}{1+u} du = \ln 2,$$

$$k_1(p) = B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2},$$

$$k_2(p) = B\left(\frac{p+2-2}{p}, \frac{q+2-2}{q}\right) = B(1,1) = 1,$$

$$\theta_3(r) = \int_0^1 \frac{1}{(1+u)^3}\left(\frac{1}{u}\right)^{-\frac{1}{\gamma}} du > \int_0^1 \frac{u}{(1+u)^3} du = \frac{1}{8},$$

$$k_3(p) = \frac{1}{2pq} B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{(p-1)\pi}{2p^2 \sin(\pi/p)}.$$

By Theorem 3.1, some corollaries are established as follows:

Corollary 3.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = 1$, $\alpha < T \leqslant \infty$ *and* $f(t), g(t) > 0$, $0 < \int_\alpha^T f^p(t)dt < +\infty$ *and* $0 < \int_\alpha^T g^q(t)dt < +\infty$ *, then we have*

$$(3.5) \quad \int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{x+y-2\alpha} dxdy < \left\{ \int_\alpha^T \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{q}} \cdot \ln 2 \right) f^p(t)dt \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_\alpha^T \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{1}{p}} \cdot \ln 2 \right) \cdot g^q(t) dt \right\}^{\frac{1}{q}} (1-r_1)^m,$$

for $T < \infty$,

and

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{x+y-2\alpha} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_\alpha^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_\alpha^\infty g^q(t)dt \right)^{\frac{1}{q}} (1-\overline{r}_1)^m. \tag{3.6}$$

Remark 3.2. *When $\alpha = 0$ and $p = q = 2$, the inequality (3.6) is reduced to a result which is equivalent to inequality (3.1) in [3] after simple computations. As a result, the inequalities (3.1), (3.2) and (3.5) – (3.6) are all extensions of (3.1) in [3].*

Corollary 3.3. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < T \leqslant \infty$ and $f(t), g(t) > 0$. If*

$$0 < \int_\alpha^T \frac{1}{t-\alpha} f^p(t)\, dt < +\infty$$

and

$$0 < \int_\alpha^T \frac{1}{t-\alpha} g^q(t)\, dt < +\infty,$$

then we obtain

$$(3.7)\quad \int_\alpha^T\int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^2}dxdy < \left\{\int_\alpha^T\left(1-\frac{t-\alpha}{2(T-\alpha)}\right)\frac{1}{t-\alpha}\cdot f^p(t)dt\right\}^{\frac{1}{p}} \times\left\{\int_\alpha^T\left(1-\frac{t-\alpha}{2(T-\alpha)}\right)\frac{1}{t-\alpha}\cdot g^q(t)dt\right\}^{\frac{1}{q}}(1-r_2)^m, \quad \text{for } T<\infty,$$

and

$$(3.8)\quad \int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^2}dxdy < \left(\int_\alpha^\infty \frac{1}{t-\alpha}f^p(t)dt\right)^{\frac{1}{p}}\left(\int_\alpha^\infty \frac{1}{t-\alpha}g^q(t)dt\right)^{\frac{1}{q}}(1-\overline{r}_2)^m.$$

Corollary 3.4. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda=3$, $\alpha<T\leqslant\infty$ *and* $f(t),g(t)>0$,

$$0<\int_\alpha^T \frac{1}{(t-\alpha)^2}f^p(t)dt<+\infty,$$

$$0<\int_\alpha^T \frac{1}{(t-\alpha)^2}g^q(t)dt<+\infty,$$

then we get

$$\begin{aligned}(3.9)\quad &\int_\alpha^T\int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^3}dxdy\\ &< \left\{\int_\alpha^T\left(\frac{(p-1)\pi}{2p^2\sin(\pi/p)}-\frac{1}{8}\left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{p}}\right)\frac{1}{(t-\alpha)^2}f^p(t)dt\right\}^{\frac{1}{p}}\\ &\times\left\{\int_\alpha^T\left(\frac{(p-1)\pi}{2p^2\sin(\pi/p)}-\frac{1}{8}\left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{q}}\right)\frac{1}{(t-\alpha)^2}g^q(t)dt\right\}^{\frac{1}{q}}(1-r_3)^m\\ &\qquad T<\infty,\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad &\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^3}dxdy\\ &< \frac{(p-1)\pi}{2p^2\sin(\pi/p)}\left(\int_\alpha^\infty\frac{1}{(t-\alpha)^2}f^p(t)dt\right)^{\frac{1}{p}}\\ &\times\left(\int_\alpha^\infty\frac{1}{(t-\alpha)^2}g^q(t)dt\right)^{\frac{1}{q}}(1-\overline{r}_3)^m.\end{aligned}$$

Since $k_\lambda(2)=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$, $\theta_\lambda(2)=\frac{1}{2}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$, and $\lambda>2-\min(2,2)=0$, we also have

Corollary 3.5. *If* $p = q = 2$, $\lambda > 0$, $\alpha < T \leqslant \infty$ *and* $f(t), g(t) > 0$,

$$0 < \int_{\alpha}^{T} (t-\alpha)^{1-\lambda} f^2(t)\,dt < +\infty,$$

$$0 < \int_{\alpha}^{T} (t-\alpha)^{1-\lambda} g^2(t)\,dt < +\infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.11}$$

$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\lambda/2}\right](t-\alpha)^{1-\lambda} f^2(t)dt\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\lambda/2}\right](t-\alpha)^{1-\lambda} g^2(t)dt\right\}^{\frac{1}{2}}(1-\overline{R})^m,$$

for $T < \infty$

and

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.12}$$

$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda} f^2(t)dt\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda} g^2(t)dt\right\}^{\frac{1}{2}}(1-\widetilde{R})^m.$$

Remark 3.3. *The inequalities (3.11), (3.12) are new generalizations of (20) in [4] and improvements of the inequalities (4) and (12) in [6] respectively.*

References

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1952.

[2] BICHENG YANG, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, **261** (2001), 295–306.

[3] MINGZHE GAO, LI TAN AND L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229** (1999), 682–689.

[4] MINGZHE GAO, On the Hilbert inequality, *Zeitschrift für Analysis und ihre Anwendungen*, **18**(4) (1999), 1117–1122.

[5] MINGZHE GAO, SHANG RONG WEI AND LEPING HE, On the Hilbert inequality with weights, *Zeitschrift für Analysis und ihre Anwendungen*, **21**(1) (2002), 257–263.

[6] BICHENG YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1998), 778–785.