Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 54, 2006

A GENERAL L_2 INEQUALITY OF GRÜSS TYPE

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Received 06 April, 2005; accepted 27 February, 2006
Communicated by S.S. Dragomir

ABSTRACT. Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general L_2 inequality of Grüss type, which includes some existing results as special cases.

Key words and phrases: Euler-Maclaurin formula, Error estimate, Grüss type inequality.

2000 *Mathematics Subject Classification.* Primary 65D32; Secondary 41A55.

1. INTRODUCTION

Inequalities of the Grüss type have been the subject renewed research interest in the past few years. The monograph [3] has had much impact on the stream of current research in this area.

Inequalities of the Grüss type can be found in e.g. [4, 5, 6, 7, 8, 9, 10] and references therein.

Recently, N. Ujević in [10] proved the following two theorems among others.

Theorem 1.1. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function, whose derivatives $f' \in L_2[0,1]$. Then,*

$$\left|\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]-\int_0^1 f(t)dt\right| \le \frac{1}{6}\sqrt{\sigma(f')}, \tag{1.1}$$

ISSN (electronic): 1443-5756

This work is supported by NNSF (Grant No. 10275054) and Hangzhou Normal College (Grant No. 2004 XNZ 03 and No. 112).
108-05

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \left(\int_0^1 f(t)dt\right)^2. \tag{1.2}$$

Inequality (1.1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

Theorem 1.2. *Under the assumptions of Theorem 1.1, for any $x \in [0,1]$, we have*

$$\left| f(x) - \left(x - \frac{1}{2}\right)[f(1) - f(0)] - \int_0^1 f(t)dt \right| \le \frac{1}{2\sqrt{3}}\sqrt{\sigma(f')}. \tag{1.3}$$

Inequality (1.3) is sharp in the sense that the constant $1/(2\sqrt{3})$ cannot be replaced by a smaller one.

Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general L_2 inequality of the Grüss type under very natural assumptions. Our results improve and generalize some existing observations.

2. A L_2 VERSION OF GRÜSS TYPE INEQUALITY

In what follows, let f be defined on $[0,1]$,

$$||f||_2 = \left(\int_0^1 f(t)dt\right)^2$$

and $L_2[0,1] = \{f| \; ||f||_2 < \infty\}$.

Some more notations and the following lemmas are needed before we proceed. In the rest of the paper, a standing assumption is that $x \in [0,1]$, n is a positive integer and $0 = t_0 < t_1 < \cdots < t_n = 1$ is an equidistant subdivision of the interval $[0,1]$ such that $t_{i+1} - t_i = h = 1/n,\ i = 0, 1, \ldots, n-1$.

We start with the following lemma.

Lemma 2.1 ([1], cf. [11])**.** *Let $f : [0,1] \to \mathbb{R}$ be such that its $(k-1)$th derivative $f^{(k-1)}$ is absolutely continuous for some positive integer k. Then for any $x \in [0,1]$, we have the Euler-Maclaurin formula*

$$\int_0^1 f(t)dt = Q_k(f,x) + E_k(Q_k; f, x), \tag{2.1}$$

where

$$\begin{aligned} Q_k(f,x) &= h\sum_{i=0}^{n-1} f(t_i + xh) - \sum_{\nu=1}^{k} \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!} B_\nu(x)h^\nu, \\ E_k(Q_k; f, x) &= \frac{h^k}{k!}\int_0^1 \widetilde{B}_k(x - nt) f^{(k)}(t)dt, \end{aligned} \tag{2.2}$$

and $\widetilde{B}_k(t) := B_k(t - \lfloor t \rfloor)$ where $B_k(t)$ is the kth Bernoulli polynomial.

Lemma 2.2. *For any $x, y \in [0,1]$, we have*

$$\int_0^1 \widetilde{B}_k(x-t)\widetilde{B}_k(y-t)dt = \frac{(-1)^{k-1}(k!)^2}{(2k)!}\widetilde{B}_{2k}(x-y). \tag{2.3}$$

Proof. We use a technique of [2]. Setting $n = 1$, $t_0 = 0$ and

$$f(t) = \frac{(-1)^k k!}{(2k)!} \widetilde{B}_{2k}(x-t),$$

in (2.1), then we have

$$f^{(k)}(t) = \widetilde{B}_k(x-t).$$

By the periodicity of $\widetilde{B}_{2k}(t)$ and the property of $B_{2k}(t)$ (see e.g. [12]), we can easily get

$$\int_0^1 \widetilde{B}_{2k}(x-t)dt = \int_0^1 B_{2k}(t)dt = 0. \tag{2.4}$$

Then we have

$$\int_0^1 f(t)dt = 0. \tag{2.5}$$

From (2.2) and the periodicity of this special function f, we have for any $y \in [0,1]$

$$\begin{aligned} Q_k(f,y) &= f(y) - \sum_{\nu=1}^{k} \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!} B_\nu(y) = f(y), \\ E_k(Q_k; f, y) &= \frac{1}{k!} \int_0^1 \widetilde{B}_k(y-t)\widetilde{B}_k(x-t)dt. \end{aligned} \tag{2.6}$$

Now from (2.1), (2.5) and (2.6), (2.3) follows. □

By Lemmas 2.1 and 2.2, we have

Corollary 2.3. *Suppose the conditions in Lemma 2.1 hold, then we have*

$$E_k(Q_k; f, x) \le h^k c_k(2) \|f^{(k)}\|_2, \tag{2.7}$$

where

$$c_k(2) = \sqrt{\frac{(-1)^{k-1}}{(2k)!} B_{2k}}.$$

Remark 2.4. From Corollary 2.3, the right side of (2.7) is independent of x.

Lemma 2.5 ([1], cf. [11])**.** *Suppose that the following quadrature rule*

$$\int_0^1 f(t)dt = \sum_{j=0}^{m-1} p_j f(x_j) \tag{2.8}$$

is exact for any polynomial of degree $\le k-1$ for some positive integer k. Let $f : [0,1] \to \mathbb{R}$ be such that its $(k-1)$th derivative $f^{(k-1)}$ is absolutely continuous. Then we have

$$\int_0^1 f(t)dt = \overline{Q}(f) + E_k(\overline{Q}; f), \tag{2.9}$$

where

$$\begin{aligned} \overline{Q}(f) &= h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h), \\ E_k(\overline{Q}; f) &= \frac{h^k}{k!} \int_0^1 g_k(nt) f^{(k)}(t)dt, \end{aligned} \tag{2.10}$$

and

$$g_k(t) = \sum_{j=0}^{m-1} p_j(\widetilde{B}_k(x_j - t) - B_k(x_j)). \tag{2.11}$$

By the Hölder inequality, we have

$$|E_k(\overline{Q}; f)| \leq \overline{c}_k(2)||f^{(k)}||_2, \tag{2.12}$$

where

$$\overline{c}_k(2) = \frac{h^k}{k!}||g_k||_2. \tag{2.13}$$

Remark 2.6. It is easy to see that (2.12) is sharp in the sense that the constant $\overline{c}_k(2)$ cannot be replaced by a smaller one.

We are now able to find an explicit expression for $\overline{c}_k(2)$.

Theorem 2.7. *Suppose that the quadrature rule* (2.3) *is exact for any polynomial of degree* $\leq k-1$ *for some positive integer* k. *Then the following equality is valid.*

$$\overline{c}_k(2) = \frac{h^k}{k!}\left\{\sum_{i,j=0}^{m-1} p_i p_j \left(\frac{(-1)^{k-1}(k!)^2}{(2k)!}\widetilde{B}_{2k}(x_i - x_j) + B_k(x_i)B_k(x_j)\right)\right\}^{\frac{1}{2}}. \tag{2.14}$$

Proof. A straightforward computation on using (2.4) and Lemma 2.2 gives

$$\begin{aligned}
\|g_k\|_2^2 &= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\widetilde{B}_k(x_i - t) - B_k(x_i)\right)\left(\widetilde{B}_k(x_j - t) - B_k(x_j))\right) dt \\
&= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\widetilde{B}_k(x_i - t)\widetilde{B}_k(x_j - t) + B_k(x_i)B_k(x_j)\right) dt \\
&= \frac{(-1)^{k-1}(k!)^2}{(2k)!}\sum_{i,j=0}^{m-1} p_i p_j \widetilde{B}_{2k}(x_i - x_j) + \frac{1}{k!}\sum_{i,j=0}^{m-1} p_i p_j B_k(x_i)B_k(x_j),
\end{aligned}$$

which in combination with (2.13) proves the conclusion as desired. □

3. EXAMPLES

Example 3.1. For the Trapezoid rule, $m = 2, x_0 = 0, x_1 = 1, p_0 = p_1 = 1/2$. It is well known that the Trapezoid rule has degree of precision 1 ($k = 2$). A direct calculation using (2.14) yields

$$\overline{c}_1(2) = \frac{h}{2\sqrt{3}}, \qquad \overline{c}_2(2) = \frac{h^2}{2\sqrt{30}}.$$

If f is absolutely continuous, then we can obtain

$$\left|\frac{1}{2}[f(0) + f(1)] - \int_0^1 f(t)dt\right| \leq \frac{1}{2\sqrt{3}}||f'||_2. \tag{3.1}$$

Replacing $f(t)$ by $f(t) - t\int_0^1 f(t)dt$ in (3.1), we get

$$\left|\frac{1}{2}[f(0) + f(1)] - \int_0^1 f(t)dt\right| \leq \frac{1}{2\sqrt{3}}\sqrt{\sigma(f')}, \tag{3.2}$$

since the Trapezoid rule has degree of precision 1.

Example 3.2. Consider the following quadrature rule

$$\int_0^1 f(t)dt = \left(x - \frac{1}{2}\right) f(0) + f(x) - \left(x - \frac{1}{2}\right) f(1), \quad x \in [0,1], \tag{3.3}$$

which has degree of precision 1 $(k = 2)$. A direct calculation using Corollary 2.3 gives

$$c_1(2) = \frac{1}{2\sqrt{3}},$$

from which and the similar argument of Example 3.1, follows (1.3).

Example 3.3. For Simpson's rule, $m = 3$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $p_0 = p_2 = 1/6$, $p_1 = 2/3$. It is well known that Simpson's rule has degree of precision 3 $(k = 4)$. A direct calculation leads to the following.

$$\bar{c}_1(2) = \frac{h}{6}; \quad \bar{c}_2(2) = \frac{h^2}{12\sqrt{30}};$$
$$\bar{c}_3(2) = \frac{h^3}{48\sqrt{105}}; \quad \bar{c}_4(2) = \frac{h^4}{576\sqrt{14}}.$$

The inequality (2.12) in combination with $\bar{c}_1(2) = h/6$ yields

$$\left| \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)dt \right| \le \frac{1}{6}||f'||_2.$$

Again replacing $f(t)$ by $f(t) - t\int_0^1 f(t)dt$ in the above inequality, we easily get Theorem 1.1.

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