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THE INEQUALITIES $G \leq L \leq I \leq A$ IN n VARIABLES

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ABSTRACT. In the short note, the inequalities $G \leq L \leq I \leq A$ for the geometric, logarithmic, identric, and arithmetic means in n variables are proved.

Key words and phrases: Inequality, Arithmetic mean, Geometric mean, Logarithmic mean, Identric mean, n variables, Van der Monde determinant.

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1. INTRODUCTION

For positive numbers a_i, $1 \leq i \leq 2$, let

$$A = A(a_1, a_2) = \frac{a_1 + a_2}{2}; \tag{1.1}$$

$$G = G(a_1, a_2) = \sqrt{a_1 a_2}\,; \tag{1.2}$$

$$I = I(a_1, a_2) = \begin{cases} \exp\left[\dfrac{a_2 \ln a_2 - a_1 \ln a_1}{a_2 - a_1} - 1\right], & a_1 < a_2, \\ a_1, & a_1 = a_2; \end{cases} \tag{1.3}$$

$$L = L(a_1, a_2) = \begin{cases} \dfrac{a_2 - a_1}{\ln a_2 - \ln a_1}, & a_1 < a_2, \\ a_1, & a_1 = a_2. \end{cases} \tag{1.4}$$

These are respectively called the arithmetic, geometric, identric, and logarithmic means.

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The logarithmic mean [1, 3], somewhat supprisingly, has applications to the economical index analysis. K.B. Stolarsky first introduced the identric mean I and proved $G \leq L \leq I \leq A$ in two variables in [4]. See [2] also.

The purpose of this short note is to prove the inequalities $G \leq L \leq I \leq A$ of the geometric, logarithmic, identric, and arithmetic means in n variables.

2. DEFINITIONS AND THE MAIN RESULT

Let $a = (a_1, a_2, \dots, a_n)$ and $a_i > 0$ for $1 \leq i \leq n$, then the arithmetic, geometric, identric, and logarithmic means in n variables are defined respectively as follows

$$A = A(a) = A(a_1, \dots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{2.1}$$

$$G = G(a) = G(a_1, \dots, a_n) = \sqrt[n]{a_1 a_2 \cdots a_n}\,, \tag{2.2}$$

$$I = I(a) = I(a_1, \dots, a_n) = \exp\left[\frac{1}{V(a)} \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a) \ln a_i - m\right], \tag{2.3}$$

$$L = L(a) = L(a_1, \dots, a_n) = \frac{(n-1)!}{V(\ln a)} \sum_{i=1}^{n} (-1)^{n+i} a_i V_i(\ln a), \tag{2.4}$$

where $\ln a = (\ln a_1, \dots, \ln a_n)$, $a_i \neq a_j$ for $i \neq j$,

$$V(a) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (a_i - a_j) \tag{2.5}$$

is the determinant of Van der Monde's matrix of the n-th order,

$$V_i(a) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_{i-1} & a_{i+1} & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_{i-1}^2 & a_{i+1}^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^{n-2} & a_2^{n-2} & \dots & a_{i-1}^{n-2} & a_{i+1}^{n-2} & \dots & a_n^{n-2} \end{vmatrix}, \tag{2.6}$$

$m = \sum_{k=1}^{n-1} \frac{1}{k}$, and $1 \leq i \leq n$.

The main result of this short note can be stated as

Theorem 2.1. *Let $a = (a_1, a_2, \dots, a_n)$ and $a_i > 0$ for $1 \leq i \leq n$, then the inequalities*

$$G(a) \leq L(a) \leq I(a) \leq A(a) \tag{2.7}$$

of the geometric, logarithmic, identric, and arithmetic means in n variables hold. The equalities in (2.7) *are valid if and only if $a_1 = a_2 = \cdots = a_n$.*

3. PROOF OF THEOREM 2.1

To prove inequalities in (2.7), we introduce the following means

$$I_r(a) = \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1\\ i_1,i_2,\ldots,i_n\ge 1}} \left[\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k\right]^{\frac{1}{\binom{n+r-2}{r-1}}}, \tag{3.1}$$

$$I'_r(a) = \prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n\ge 0}} \left[\sum_{k=1}^{n} \frac{i_k}{r} a_k\right]^{\frac{1}{\binom{n+r-1}{r}}}, \tag{3.2}$$

$$L_r(a) = \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n\ge 0}} \prod_{k=1}^{n} a_k^{i_k/r}, \tag{3.3}$$

$$L'_r(a) = \frac{1}{\binom{n+r-2}{r-1}} \sum_{\substack{i_1+i_2+\cdots+i_n=n+r-1\\ i_1,i_2,\ldots,i_n\ge 1}} \prod_{k=1}^{n} a_k^{i_k/(n+r-1)}, \tag{3.4}$$

where $a = (a_1, a_2, \ldots, a_n)$ and $a_i > 0$ for $1 \le i \le n$.

Lemma 3.1. *Let $a = (a_1, a_2, \ldots, a_n)$ and $a_i > 0$ for $1 \le i \le n$, then we have*

(1) $I_1(a) = L_1(a) = A(a)$;
(2) $I'_1(a) = L'_1(a) = G(a)$;
(3) *For* $1 \le j \le n-1$ *and*

$$\pi_j = \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1\\ i_{k_1}=i_{k_2}=\cdots=i_{k_j}=0,\ \text{for the rest } i_k\ge 1}} \sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k, \tag{3.5}$$

we have

$$I'_{n+r-1}(a) = \prod_{j=1}^{n-1} \pi_j^{1/\binom{2n+r-2}{n+r-1}} \left[I_r(a)\right]^{\binom{n+r-2}{r-1}/\binom{2n+r-2}{n+r-1}}; \tag{3.6}$$

(4) *For* $1 \le j \le n-1$ *and*

$$\delta_j = \sum_{\substack{i_1+i_2+\cdots+i_n=n+r-1\\ i_{k_1}=i_{k_2}=\cdots=i_{k_j}=0,\ \text{for the rest } i_k\ge 1}} \prod_{k=1}^{n} a_k^{i_k/(n+r-1)}, \tag{3.7}$$

we have

$$L_{n+r-1}(a) = \frac{1}{\binom{2n+r-2}{n+r-1}} \left[\sum_{j=1}^{n-1} \delta_j + \binom{n+r-2}{r-1} L'_r(a)\right]; \tag{3.8}$$

(5) *If* $r \in \mathbb{N}$, *then*
  (a) $I_r(a) \ge I_{r+1}(a)$,
  (b) $I'_r(a) \le I'_{r+1}(a)$,
  (c) $L_r(a) \ge L_{r+1}(a)$,
  (d) $L'_r(a) \le L'_{r+1}(a)$,
  (e) $I_r(a) \ge L_r(a)$,
  (f) $I'_r(a) \ge L'_r(a)$,

where equalities above hold if and only if $a_1 = a_2 \cdots = a_n$.

Proof. The formula (3.6) follows from standard arguments and formulas (3.1) and (3.2).

If $r \in \mathbb{N}$ and $i_1 + i_2 + \cdots + i_n = n + r - 1$, then

$$\begin{aligned}
\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k &= \sum_{k=1}^{n} \frac{i_k}{n+r} \cdot \frac{n+r}{n+r-1} a_k \\
&= \frac{n+r}{n+r-1} \sum_{k=1}^{n} \frac{i_k}{n+r} a_k \\
&= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j + 1 \right] \sum_{k=1}^{n} \frac{i_k}{n+r} a_k \\
&= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j \sum_{k=1}^{n} \frac{i_k}{n+r} a_k + \sum_{k=1}^{n} \frac{i_k}{n+r} a_k \right] \\
&= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j \sum_{k=1}^{n} \frac{i_k}{n+r} a_k + \sum_{j=1}^{n} \frac{i_j}{n+r} a_j \right] \\
&= \sum_{j=1}^{n} \frac{i_j}{n+r-1} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right].
\end{aligned}$$

By using the weighted arithmetic-geometric mean inequality, we have

$$\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k \geq \prod_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)},$$

and then

$$\begin{aligned}
\prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k \quad (3.9)\\
\geq \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \prod_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)} \\
= \prod_{j=1}^{n} \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)} \\
= \prod_{j=1}^{n} \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_{j-1},\nu_{j+1},\ldots,\nu_n \geq 1; \nu_j \geq 2}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{(\nu_j-1)/(n+r-1)} \\
= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n \geq 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{\sum_{j=1}^{n}(\nu_j-1)/(n+r-1)}
\end{aligned}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k\right]^{\left(\sum_{j=1}^{n}\nu_j - n\right)/(n+r-1)}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k\right]^{r/(n+r-1)}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k\right]^{\binom{n+r-2}{r-1}/\binom{n+r-1}{r}},$$

notice that the result from line 4 to line 5 in (3.9) follows from a simple fact that

$$\left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k\right]^{(\nu_j-1)/(n+r-1)} = 1 \text{ for } \nu_j = 1.$$

The equalities above are valid if and only if

$$\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_1}{n+r} = \sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_2}{n+r} = \cdots = \sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_n}{n+r},$$

which is equivalent to $a_1 = a_2 = \cdots = a_n$. This implies that $I_r(a) \ge I_{r+1}(a)$.

The inequality $I_r(a) \ge L_r(a)$ follows easily from the generalized Hölder inequality

$$\prod_{\substack{i_1+i_2+\cdots+i_n=r+1 \\ i_1\ge 1, i_2\ge 1,\ldots,i_n\ge 1}} \left[\sum_{j=1}^{n} i_j a_j\right]^{\frac{1}{r}} \ge \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\ge 0, i_2\ge 0,\ldots,i_n\ge 0}} \left[\prod_{j=1}^{n} a_j^{i_j}\right]^{\frac{1}{r}}. \tag{3.10}$$

The proofs of other formulas and inequalities will be left to the readers. □

Lemma 3.2. *Let* $a = (a_1, a_2, \ldots, a_n)$ *and* $a_i > 0$ *for* $1 \le i \le n$*, then we have*

(1) $\lim_{r\to\infty} I_r(a) = \lim_{r\to\infty} I'_r(a) = I(a)$,
(2) $\lim_{r\to\infty} L_r(a) = \lim_{r\to\infty} L'_r(a) = L(a)$.

Proof. It is easy to see that $\lim_{r\to\infty} I_r(a) = \lim_{r\to\infty} I'_r(a)$ and $\lim_{r\to\infty} L_r(a) = \lim_{r\to\infty} L'_r(a)$, since

$$\lim_{r\to\infty} \pi_j^{1/\binom{2n+r-2}{n+r-1}} = 1, \quad \lim_{r\to\infty} \frac{1}{\binom{2n+r-2}{n+r-1}} \sum_{j=1}^{n-1} \delta_j = 0, \quad \lim_{r\to\infty} \frac{\binom{n+r-2}{r-1}}{\binom{2n+r-2}{n+r-1}} = 1.$$

Straightforward computation yields

$$\begin{aligned}
\ln \lim_{r\to\infty} I_r'(a) &= \lim_{r\to\infty} \ln I_r'(a) \\
&= \lim_{r\to\infty} \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n\geq 0}} \ln \sum_{k=1}^{n} \frac{i_k}{r} a_k \\
&= (n-1)! \int\cdots\int_{\substack{x_1+x_2+\cdots+x_{n-1}\leq 1 \\ x_\ell\geq 0, 1\leq \ell\leq n-1}} \ln\left[\left(1-\sum_{i=1}^{n-1} x_i\right) a_1 + \sum_{j=2}^{n} x_{j-1}a_j\right] dx_1 dx_2 \ldots dx_{n-1} \\
&= \frac{(n-1)!}{V(a)} \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a)(\ln a_i - m) \\
&= \ln I(a)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{r\to\infty} L_r(a) &= \lim_{r\to\infty} \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n\geq 0}} \prod_{k=1}^{n} a_k^{i_k/r} \\
&= (n-1)! \int\cdots\int_{\substack{x_1+x_2+\cdots+x_{n-1}\leq 1 \\ x_\ell\geq 0, 1\leq \ell\leq n-1}} a_1^{1-\sum_{i=1}^{n-1} x_i} a_2^{x_1} \cdots a_n^{x_{n-1}} dx_1 dx_2 \cdots dx_{n-1} \\
&= \frac{(n-1)!}{V(\ln a)} \sum_{i=1}^{n} (-1)^{n+i} a_i V_i(\ln a) \\
&= L(a).
\end{aligned} \tag{3.11}$$

The proof is complete. □

Proof of Theorem 2.1. This follows from combination of Lemma 3.1 and Lemma 3.2. □

REFERENCES

[1] E. LEACH AND M. SHOLANDER, Extended mean values, *Amer. Math. Monthly*, **85** (1978), 84–90.

[2] A.O. PITTENGER, Two logarithmic mean in n variables, *Amer. Math. Monthly*, **92** (1985), 99–104.

[3] G. PÓLYA AND G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.

[4] K.B. STOLARSKY, Generalizations of the logarithmic mean, *Mag. Math.*, **48** (1975), 87–92.