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PARTIAL SUMS OF CERTAIN MEROMORPHIC $p-$VALENT FUNCTIONS

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Abstract

In this paper we establish some results concerning the partial sums of meromorphic p-valent starlike functions and meromorphic p-valent convex functions.

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Contents

1. Introduction

Let $\sum(p)$ $(p \in \mathbb{N} = \{1, 2, \dots\})$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p-1} z^{k+p-1} \qquad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and $p-$valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. A function $f(z)$ in $\sum(p)$ is said to belong to $\sum^*(p, \alpha)$, the class of meromorphically p-valent starlike functions of order α $(0 \leq \alpha < p)$, if and only if

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (0 \leq \alpha < p; z \in U = U^* \cup \{0\}). \tag{1.2}$$

A function $f(z)$ in $\sum(p)$ is said to belong to $\sum_k(p, \alpha)$, the class of $p-$valent convex functions of order $\alpha (0 \leq \alpha < p)$, if and only if

$$-\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (0 \leq \alpha < p; z \in U). \tag{1.3}$$

It follows from (1.2) and (1.3) that

$$f(z) \in \sum\nolimits_k(p, \alpha) \Longleftrightarrow -\frac{zf'(z)}{p} \in \sum\nolimits^*(p, \alpha). \tag{1.4}$$

The classes $\sum^*(p, \alpha)$and $\sum_k(p, \alpha)$ were studied by Kumar and Shukla [6]. A sufficient condition for a function $f(z)$ of the form (1.1) to be in $\sum^*(p, \alpha)$ is that

$$\sum_{k=1}^{\infty} (k + p - 1 + \alpha) |a_{k+p-1}| \leq (p - \alpha) \tag{1.5}$$

and to be in $\sum_k(p,\alpha)$ is that

$$\sum_{k=1}^{\infty}\binom{k+p-1}{p}(k+p-1+\alpha)\left|a_{k+p-1}\right|\leq(p-\alpha). \tag{1.6}$$

Further, we note that these sufficient conditions are also necessary for functions of the form (1.1) with positive or negative coefficients (see [1], [2], [5], [9], [14] and [15]). Recently , Silverman [11] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also, Li and Owa [7] obtained the sharp radius which for the normalized univalent functions in U, the partial sums of the well known Libera integral operator [8] imply starlikeness. Further , for various other interesting developments concerning partial sums of analytic univalent functions (see [3], [10], [12], [13] and [16]).

Recently , Cho and Owa [4] have investigated the ratio of a function of the form (1.1) (with $p=1$) to its sequence of partial sums $f_n(z)=\frac{1}{z}+\sum_{k=1}^{n}a_kz^k$ when the coefficients are sufficiently small to satisfy either condition (1.5) or (1.6) with $p=1$. Also Cho and Owa [4] have determined sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\}$, and $\operatorname{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\}$.

In this paper, applying methods used by Silverman [11] and Cho and Owa [4], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums

$$f_{n+p-1}(z)=\frac{1}{z^p}+\sum_{k=1}^{n+p-1}a_{k+p-1}z^{k+p-1}$$

when the coefficients are sufficiently small to satisfy either condition (1.5) or

(1.6). More precisely, we will determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f'(z)}{f'_{n+p-1}(z)}\right\}$, and $\operatorname{Re}\left\{\frac{f'_{n+p-1}(z)}{f'(z)}\right\}$.

In the sequel, we will make use of the well-known result that $\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0$ $(z \in U)$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|w(z)| \leq |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by

$$f_{n+p-1}(z) = \frac{1}{z^p} + \sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+p-1}.$$

2. Main Results

Theorem 2.1. *If f of the form (1.1) satisfies condition (1.5), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\} \geq \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad (z \in U). \tag{2.1}$$

The result is sharp for every n and p, with extremal function

$$f(z) = \frac{1}{z^p} + \frac{p-\alpha}{n+2p-1+\alpha} z^{n+2p-1} \qquad (n \geq 0; p \in \mathbb{N}). \tag{2.2}$$

Proof. We may write

$$\begin{aligned}
&\frac{n+2p-1+\alpha}{p-\alpha}\left[\frac{f(z)}{f_{n+p-1}(z)} - \frac{n+p-1+2\alpha}{n+2p-1+\alpha}\right] \\
&= \frac{1+\sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{1+\sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1}} \\
&= \frac{1+A(z)}{1+B(z)}.
\end{aligned}$$

Set $\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}$, so that $w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$w(z) = \frac{\left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{2+2\sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}$$

and

$$|w(z)| \leq \frac{\left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|}{2-2\sum_{k=1}^{n+p-1}|a_{k+p-1}|-\left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}| \leq 2-2\sum_{k=1}^{n+p-1}|a_{k+p-1}|,$$

which is equivalent to

$$\sum_{k=1}^{n+p-1}|a_{k+p-1}| + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}| \leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by $\sum_{k=1}^{\infty}\left(\frac{k+p-1+\alpha}{p-\alpha}\right)|a_{k+p-1}|$, which is equivalent to

$$\sum_{k=1}^{n+p-1}\left(\frac{k+2\alpha-1}{p-\alpha}\right)|a_{k+p-1}| + \sum_{k=n+p}^{\infty}\left(\frac{k-n-p}{p-\alpha}\right)|a_{k+p-1}| \geq 0.$$

To see that the function f given by (2.2) gives the sharp result, we observe for $z = re^{\pi i/(n+3p-1)}$ that

$$\begin{aligned}\frac{f(z)}{f_{n+p-1}(z)} &= 1+\frac{p-\alpha}{n+2p-1+\alpha}z^{n+3p-1} \to 1-\frac{p-\alpha}{n+2p-1+\alpha}\\ &= \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad \text{when } r \to 1^{-}.\end{aligned}$$

Therefore we complete the proof of Theorem 2.1. $\square$

Theorem 2.2. *If f of the form (1.1) satisfies condition (1.6), then*

$$\begin{aligned}(2.4)\quad \operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\} \\ \geq \frac{(n+2p)(n+2p-2+\alpha)+(1-p)(1+p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)} \qquad (z\in U).\end{aligned}$$

The result is sharp for every n and p, with extremal function

$$(2.5)\quad f(z)=\frac{1}{z^p}+\frac{p(p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)}z^{n+2p-1} \qquad (n\geq 0; p\in\mathbb{N}).$$

Proof. We write

$$\begin{aligned}&\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \\ &\qquad\times\left[\frac{f(z)}{f_{n+p-1}(z)}-\frac{(n+2p)(n+2p-2+\alpha)+(1-p)(1+p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)}\right] \\ &=\frac{1+\sum_{k=1}^{n+p-1}a_{k+p-1}z^{k+2p-1}+\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty}a_{k+p-1}z^{k+2p-1}}{1+\sum_{k=1}^{n+p-1}a_{k+p-1}z^{k+2p-1}} \\ &=\frac{1+w(z)}{1-w(z)},\end{aligned}$$

where

$$w(z) = \frac{\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum\limits_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{2 + 2\sum\limits_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} + \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum\limits_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}.$$

Now

$$|w(z)| \leq \frac{\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum\limits_{k=n+p}^{\infty} |a_{k+p-1}|}{2 - 2\sum\limits_{k=1}^{n+p-1} |a_{k+p-1}| - \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum\limits_{k=n+p}^{\infty} |a_{k+p-1}|} \leq 1,$$

if

$$\sum_{k=1}^{n+p-1} |a_{k+p-1}| + \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum_{k=n+p}^{\infty} |a_{k+p-1}| \leq 1. \tag{2.6}$$

The left hand side of (2.6) is bounded above by

$$\sum_{k=1}^{\infty} \frac{(k+p-1)(k+p-1+\alpha)}{p(p-\alpha)} |a_{k+p-1}|$$

if

$$\frac{1}{p(p-\alpha)}\left\{\sum_{k=1}^{n+p-1}\left[(k+p-1)(k+p-1+\alpha)-p(p-\alpha)\right]|a_{k+p-1}|\right.$$
$$+\sum_{k=n+p}^{\infty}\left[(k+p-1)(k+p-1+\alpha)\right.$$
$$\left.\left.-(n+2p-1)(n+2p-1+\alpha)\right]|a_{k+p-1}|\right\}\geq 0,$$

and the proof is completed. □

We next determine bounds for $\operatorname{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\}$.

Theorem 2.3.

(a) If f of the form (1.1) satisfies condition (1.5), then

$$\operatorname{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\}\geq\frac{n+2p-1+\alpha}{n+3p-1}\qquad(z\in U). \tag{2.7}$$

(b) If f of the form (1.1) satisfies condition (1.6), then

$$\operatorname{Re}\left\{\frac{f_{n+p-1)}(z)}{f(z)}\right\}$$
$$\geq\frac{(n+2p-1)(n+2p-1+\alpha)}{(n+2p-1)(n+2p)-n(1-\alpha)+(1-p)(1-p-\alpha)}\qquad(z\in U). \tag{2.8}$$

Equalities hold in (a) and (b) for the functions given by (2.2) and (2.5), respectively.

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\begin{aligned}
&\frac{(n+2p-1)}{(p-\alpha)}\left[\frac{f_{n+p-1)}(z)}{f(z)}-\frac{n+2p-1+\alpha}{n+3p-1}\right] \\
&= \frac{1+\sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} - \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{1+\sum_{k=1}^{\infty} a_{k+p-1} z^{k+2p-1}} \\
&= \frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{n+3p-1}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|}{2-2\sum_{k=1}^{n+p-1}|a_{k+p-1}| - \left(\frac{n+p-1+2\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n+p-1}|a_{k+p-1}| + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}| \leq 1. \tag{2.9}$$

Since the left hand side of (2.9) is bounded above by $\sum_{k=1}^{\infty}\frac{(n+p-1+\alpha)}{(p-\alpha)}|a_{k+p-1}|$, the proof is completed. □

We next turn to ratios involving derivatives.

Theorem 2.4. *If f of the form (1.1) satisfies condition (1.5), then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_{n+p-1}(z)}\right\} \geq \frac{2p(n+2p-1)-\alpha(n+p-1)}{p(n+2p-1+\alpha)} \qquad (z \in U), \tag{2.10}$$

$$\operatorname{Re}\left\{\frac{f'_{n+p-1}(z)}{f'(z)}\right\} \geq \frac{p(n+2p-1+\alpha)}{\alpha(n+3p-1)} \qquad (z \in U; \alpha \neq 0). \tag{2.11}$$

The extremal function for the case (2.10) *is given by* (2.2) *and the extremal function for the case* (2.11) *is given by* (2.2) *with* $\alpha \neq 0$.

The proof of Theorem 2.4 follows the pattern of those in Theorem 2.1 and (a) of Theorem 2.3 and so the details may be omitted.

Remark 1. *Putting $p = 1$ in Theorem 2.4, we obtain the following corollary:*

Corollary 2.5. *If f of the form (1.1) (with $p = 1$) satisfies condition (1.5) (with $p = 1$), then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_{n}(z)}\right\} \geq \frac{2(n+1)-\alpha n}{n+1+\alpha} \qquad (z \in U), \tag{2.12}$$

$$\operatorname{Re}\left\{\frac{f'_{n}(z)}{f'(z)}\right\} \geq \frac{n+1+\alpha}{\alpha(n+2)} \qquad (z \in U; \alpha \neq 0). \tag{2.13}$$

The extremal function for the case (2.12) is given by (2.2) (with $p = 1$) and the extremal function for the case (2.13) is given by (2.2) (with $p = 1$ and $\alpha \neq 0$).

Remark 2. *We note that Corollary 2.5 corrects the result obtained by Cho and Owa [4, Theorem 5].*

Theorem 2.6. *If f of the form* (1.1) *satisfies condition* (1.6)*, then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_{n+p-1}(z)}\right\} \geq \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad (z \in U), \tag{2.14}$$

$$\operatorname{Re}\left\{\frac{f'_{n+p-1}(z)}{f'(z)}\right\} \geq \frac{n+2p-1+\alpha}{n+3p-1} \qquad (z \in U). \tag{2.15}$$

In both cases, the extremal function is given by (2.5).

Proof. It is well known that $f \in \sum_k(p,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum^*(p,\alpha)$. In particular, f satisfies condition (1.6) if and only if $-\frac{zf'(z)}{p}$ satisfies condition (1.5). Thus, (2.14) is an immediate consequence of Theorem 2.1 and (2.15) follows directly from Theorem 2.3 (a). □

Remark 3. *Putting $p = 1$ in the above results we get the results obtained by Cho and Owa [4].*

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