Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 1, Article 10, 2004

ON ESTIMATES OF NORMAL STRUCTURE COEFFICIENTS OF BANACH SPACES

Y. Q. YAN

DEPARTMENT OF MATHEMATICS, SUZHOU UNIVERSITY
SUZHOU, JIANGSU, P.R. CHINA, 215006.
yanyq@pub.sz.jsinfo.net

Received 22 August, 2003; accepted 09 January, 2004
Communicated by C.P. Niculescu

ABSTRACT. We obtained the estimates of Normal structure coefficient $N(X)$ by Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X and found that X has uniform normal structure if $C_{NJ}(X) < (3+\sqrt{5})/4$. These results improved both Prus' [6] and Kato, Maligranda and Takahashi's [4] work.

Key words and phrases: Normal structure coefficient, Neumann-Jordan constant, Non-square constants, Banach space.

2000 *Mathematics Subject Classification.* 46B20, 46E30.

1. INTRODUCTION

Let $X = (X, \|\cdot\|)$ be a real Banach space. Geometrical properties of a Banach space X are determined by its unit ball $B_X = \{x \in X : \|x\| \le 1\}$ or its unit sphere $S_X = \{x \in X : \|x\| = 1\}$. A Banach space X is called uniformly non-square if there exists a $\delta \in (0,1)$ such that for any $x, y \in S_X$ either $\|x+y\|/2 \le 1-\delta$ or $\|x-y\|/2 \le 1-\delta$. The constant

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

is called the non-square constant of X in the sense of James. It is well-known that $\sqrt{2} \le J(X) \le 2$ if $\dim X \ge 2$. The Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \text{ not both zero}\right\}.$$

Clearly, $1 \le C_{NJ}(X) \le 2$. and X is a Hilbert space if and only if $C_{NJ}(X) = 1$. Kato, Maligranda and Takahashi [4] proved that for any non-trivial Banach space X ($\dim X \ge 2$),

$$\frac{1}{2}J(X)^2 \le C_{NJ}(X) \le \frac{J(X)^2}{(J(X)-1)^2+1}. \tag{1.1}$$

ISSN (electronic): 1443-5756

114-03

A Banach space X is said to have normal structure if $r(K) < \text{diam}(K)$ for every non-singleton closed bounded convex subset K of X, where $\text{diam}(K) = \sup\{\|x - y\| : x, y \in K\}$ is the diameter of K and $r(K) = \inf\{\sup\{\|x-y\| : y \in K\} : x \in K\}$ is the Chebyshev radius of K. The normal structure coefficient of X is the number

$$N(X) = \inf\{\text{diam}(K)/r(K) : K \subset X \text{ bounded and convex}, \text{diam}(K) > 0\}.$$

Obviously, $1 \le N(X) \le 2$. It is known [5], [2] that if the space X is reflexive, then the infimum in the definition of $N(X)$ can be taken over all convex hulls of finite subsets of X. The space X is said to have uniform normal structure if $N(X) > 1$. If X has uniform normal structure, then X is reflexive and hence X has fixed point property. Gao and Lau [3] showed that if $J(X) < 3/2$, then X has uniform normal structure. Prus [6] gave an estimate of $N(X)$ by $J(X)$ which contains Gao-Lau's [3] and Bynum's [1] results: For any non-trivial Banach space X,

$$N(X) \ge J(X) + 1 - \{(J(X)+1)^2 - 4\}^{\frac{1}{2}}. \tag{1.2}$$

Kato, Maligranda and Takahashi [4] proved

$$N(X) \ge \left(C_{NJ}(X) - \frac{1}{4}\right)^{-\frac{1}{2}}, \tag{1.3}$$

which implies that if $C_{NJ}(X) < 5/4$ then X has uniform normal structure. This result is a little finer than Gao-Lau's condition by $J(X)$. This paper is devoted to improving the above results.

2. MAIN RESULTS

Our proofs are based on the idea due to Prus [6], who estimated $N(X)$ by modulus of convexity of X. Let C be a convex hull of a finite subset of a Banach space X. Since C is compact, there exists an element $y \in C$ such that $\sup\{\|x - y\| : x \in C\} = r(C)$. Translating the set C we can assume that $y = 0$. Prus [6] gave the following

Proposition 2.1. *Let C be a convex hull of a finite subset of a Banach space X such that $\sup\{\|x\| : x \in C\} = r(C)$. Then there exist points $x_1, \dots, x_n \in C$, norm-one functionals $x_1^*, \dots, x_n^* \in X^*$ and nonnegative number $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$,*

$$x_i^*(x_i) = \|x_i\| = r(C)$$

for $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_i x_i^*(x) = 0$ whenever $\lambda x \in C$ for some $\lambda > 0$.

Without loss of generality, we assume $r(C) = 1$ therefore $C \subset B_X$.

Theorem 2.2. *Let X be a non-trivial Banach space with the Neumann-Jordan constant $C_{NJ}(X)$. Then*

$$N(X) \ge \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}. \tag{2.1}$$

Proof. Let C be a convex hull of a finite subset of X such that $\sup\{\|x\| : x \in C\} = r(C) = 1$ and $\text{diam}C = d$. By Proposition 2.1 we obtain elements $x_1, \dots, x_n \in C$, norm-one functionals $x_1^*, \dots, x_n^* \in X^*$ and nonnegative numbers $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$, $x_i^*(x_i) = 1$ and $\sum_{j=1}^n \lambda_j x_j^*(x_i) = 0$ for $i = 1, \dots, n$.

Define

$$x_{i,j} = \frac{1}{d}(x_i - x_j), y_{i,j} = x_i \tag{2.2}$$

$i, j = 1, \dots, n$. Clearly $x_{i,j}, y_{i,j} \in B_X$ and $x_{i,j} + y_{i,j} = (1 + 1/d)x_i - (1/d)x_j$, $x_{i,j} - y_{i,j} = (-1 + 1/d)x_i - (1/d)x_j$. It follows that

$$\begin{aligned}
&\sum_{i,j=1}^{n} \lambda_i\lambda_j \left[\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2\right] \\
&\quad \geq \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left[x_i^*(x_{i,j} + y_{i,j})\right]^2 + \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \left[x_j^*(x_{i,j} - y_{i,j})\right]^2 \\
&\quad = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left[1 + \frac{1}{d} - \frac{1}{d}x_i^*(x_j)\right]^2 + \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \left[\frac{1}{d} + \left(1 - \frac{1}{d}\right) x_j^*(x_i)\right]^2 \\
&\quad = \left(1 - \frac{1}{d}\right)^2 - 2\left(1 - \frac{1}{d}\right)\frac{1}{d}\sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i x_i^*(x_j) + \frac{1}{d^2}\sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i [x_i^*(x_j)]^2 \\
&\qquad + \frac{1}{d^2} + 2\left(1 - \frac{1}{d}\right)\frac{1}{d}\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j x_j^*(x_i) + \left(1 - \frac{1}{d}\right)^2 \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j [x_j^*(x_i)]^2 \\
&\quad \geq \left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2}.
\end{aligned}$$

Therefore there exist i, j such that

$$\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2 \geq \left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2}.$$

From the definition of Neumann-Jordan constant we see that

$$C_{NJ}(X) \geq \frac{\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2}{4} \geq \frac{1}{4}\left[\left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2}\right]. \tag{2.3}$$

This inequality is equivalent to the following one

$$d \geq \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}. \tag{2.4}$$

Therefore, we obtain the desired estimate (2.1) since $C \subset X$ is arbitrary. The proof is finished. □

It is easy to check that

$$\frac{1}{\sqrt{C_{NJ}(X) - \frac{1}{4}}} < \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}$$

when $1 < C_{NJ}(X) < 5/4$. Therefore, the estimate of the above theorem improves (1.3). It is also not difficult to check that

$$\sqrt{2C_{NJ}(X)} + 1 - \left((\sqrt{2C_{NJ}(X)} + 1)^2 - 4\right)^{\frac{1}{2}} < \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1} \tag{2.5}$$

when $1 < C_{NJ}(X) < 5/4$. Since $J(X) \leq \sqrt{2C_{NJ}(X)}$, and the function $x+1-((x+1)^2-4)^{1/2}$ is decreasing, we have (1.2) from (2.1) and (2.5). So (1.2) becomes a corollary of (2.1).

Prus [6] gave the result that if $J(X) < 4/3$, then $N(X) > 1$. Gao and Lau [3] gave a condition that if $J(X) < 3/2$ then $N(X) > 1$. Then they asked whether the estimate $J(X) < 3/2$ is sharp for X to have uniform normal structure. Kato, Maligranda and Takahashi [4] found that if $C_{NJ}(X) < 5/4$, which implies $J(X) < \sqrt{10}/2$, then $N(X) > 1$. The following theorem will give a wider interval of $C_{NJ}(X)$ for X to have uniform normal structure.

Theorem 2.3. *Let X be a non-trivial Banach space with the Neumann-Jordan constant $C_{NJ}(X)$ and normal structure coefficient $N(X)$. Then*

$$C_{NJ}(X) \geq \frac{\left(\sqrt{\frac{N^2(X)}{4} + \frac{1}{N^2(X)}} + N(X) - \frac{1}{N(X)}\right)^2 + \frac{1}{N^2(X)}}{2\left[1 + \left(\sqrt{\frac{N^2(X)}{4} + \frac{1}{N^2(X)}} + N(X) - \frac{2}{N(X)}\right)^2\right]}. \tag{2.6}$$

Moreover, if $C_{NJ}(X) < (3+\sqrt{5})/4$ or $J(X) < (1+\sqrt{5})/2$, then $N(X) > 1$ and hence X has uniform normal structure.

Proof. We modify the first step in the proof of Theorem 2.2. In (2.2), let

$$x_{i,j} = \frac{1}{d}(x_i - x_j), y_{i,j} = tx_i \tag{2.7}$$

with $t > 0$. Then $\|x_{i,j}\| \leq 1, \|y_{i,j}\| = t$. Similar to (2.3), we obtain

$$C_{NJ}(X) \geq \frac{\left(t + \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2(1+t^2)} \tag{2.8}$$

for any $t > 0$. The function

$$f(t) = \frac{\left(t + \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2(1+t^2)}$$

reach the maximum at the point

$$t_0 = \sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}.$$

It is decreasing on $t > t_0$ and increasing on $0 < t < t_0$. Therefore, we have

$$C_{NJ}(X) \geq \frac{\left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2\left[1 + \left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}\right)^2\right]}. \tag{2.9}$$

Since the function

$$c = g(d) := \frac{\left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2\left[1 + \left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}\right)^2\right]}$$

is strictly decreasing and continuous on $1 \leq d \leq 2$, we know that the inverse function $d = g^{-1}(c)$ exists and must also be decreasing. Thus, we have from (2.9) that $d \geq g^{-1}(C_{NJ}(X))$. It follows by take the infimum of d that $N(X) \geq g^{-1}(C_{NJ}(X))$. Equivalently, we have (2.6). From the above statements of monotony property, we deduce that $N(X) = 1$ is corresponding to $C_{NJ}(X) = (3+\sqrt{5})/4$. Therefore, if $C_{NJ}(X) < (3+\sqrt{5})/4$, then $N(X) > 1$. Since the non-square constant $J(X) \leq \sqrt{2C_{NX}}$, we have in other word that if $J(X) < (1+\sqrt{5})/2$, then $N(X) > 1$. $\square$

REFERENCES

[1] W.L. BYNUM, About some parameters of normed linear spaces, *Pacific J. Math.*, **86** (1980), 427–436.

[2] T. DOMINGUEZ BENAVIDES, Normal structued coefficients of $L^p(\Omega)$, *Proc. Roy. Soc. Sect.*, **A117** (1991), 299–303.

[3] J. GAO AND K.S. LAU, On two classes of Banach spaces with uniform normal structure, *Studia Math.*, **99** (1991), 41–56.

[4] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, *Studia Math.*, **144** (2001), 275–295.

[5] E. MALUTA, Uniformly normal structure and telated coefficients, *Pacific J. Math.*, **111**(1984), 357–369.

[6] S. PRUS, Some estimates for the normal structure coefficient in Banach spaces, *Rend. Circ. Mat. Palermo*, **40** (1991), 128–135.

[7] M.M. RAO AND Z.D. REN, Applications of Orlicz spaces, *Marcel Dekker, New York*, (2002), 49–53.