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ON GENERALIZATION OF ČEBYŠEV TYPE INEQUALITIES

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ABSTRACT. A generalization of Pečarić's extension of Montgomery's identity is established and used to derive new Čebyšev type inequalities.

Key words and phrases: Montgomery and Čebyšev-Grüss type inequalities, Pečarić's extension, Montgomery identity.

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1. INTRODUCTION

In the present work we establish a generalization of Pečarić's extension of 'Montgomery's' identity and use it to derive new Čebyšev type inequalities.

We recall the Čebyšev inequality [1], given by the following:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.2}$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]} |p(t)|$.

Pachpatte in [6] established new inequalities of the Čebyšev type by using Pečarić's extension of the Montgomery identity [7].

114-06

2. STATEMENT OF RESULTS

From [3], if $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$ with the first derivative $f'(t)$ integrable on $[a,b]$, then the Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P(x,t)\,f'(t)\,dt, \tag{2.1}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\dfrac{t-a}{b-a}, & a\le t\le x\\[2ex] \dfrac{t-b}{b-a}, & x<t\le b.\end{cases}$$

We assume that $w:[a,b]\to[0,+\infty[$ is some probability density function, i.e. $\int_a^b w(t)\,dt=1$, and set $W(t)=\int_a^t w(x)\,dx$ for $a\le t\le b$, $W(t)=0$ for $t<a$ and for $t>b$. We then have the following identity given by Pečarić in [7], that is the weighted generalization of the Montgomery identity:

$$f(x)=\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt, \tag{2.2}$$

where the weighted Peano kernel P_w is:

$$P_w(x,t)=\begin{cases}W(t), & a\le t\le x\\ W(t)-1, & x<t\le b\end{cases} \tag{2.3}$$

Let $\varphi:[0,1]\to\mathbb{R}$ be a differentiable function on $[0,1]$, with $\varphi(0)=0$, $\varphi(1)\neq0$ and φ' integrable on $[0,1]$. To simplify the notation, for some given functions $w,\ f,\ g:[a,b]\to\mathbb{R}$, we set

$$\begin{aligned}T(w,f,g,\varphi')&=\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right)f(x)\,g(x)\,dx\\ &-\frac{1}{\varphi(1)}\left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right)f(x)\,dx\right]\left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right)g(x)\,dx\right].\end{aligned} \tag{2.4}$$

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable and $f'(t)$ integrable on $[a,b]$, then,*

$$f(x)=\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt+\frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\,f'(t)dt, \tag{2.5}$$

where $P_{w,\varphi}$ is a generalization of the weighted Peano kernel defined by:

$$P_{w,\varphi}(x,t)=\begin{cases}\varphi(W(t)), & a\le t\le x;\\ \varphi(W(t))-\varphi(1), & x<t\le b.\end{cases} \tag{2.6}$$

Proof. Using the hypothesis on φ,

$$\begin{aligned}
(2.7)\qquad & \int_a^b P_{w,\varphi}(x,t)\, f'(t)dt \\
&= \int_a^x \varphi(W(t))\, f'(t)dt + \int_x^b (\varphi(W(t)) - \varphi(1))\, f'(t)dt \\
&= \int_a^b \varphi(W(t))\, f'(t)dt - \varphi(1)\int_x^b f'(t)dt \\
&= [\varphi(W(t))\, f(t)]_a^b - \int_a^b w(t)\,\varphi'(W(t))\, f(t)\, dt - \varphi(1)\,[f(b) - f(x)] \\
&= \varphi(1)\, f(x) - \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt.
\end{aligned}$$

Multiplying both sides by $1/\varphi(1)$, we obtain,

$$f(x) = \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt + \frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\, f'(t)dt$$

and this completes the proof. □

Theorem 2.2. *Let f, g $[a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and f', g' be integrable on $[a,b]$ and let w, φ be as in Theorem 2.1, then,*

$$|T(w,f,g,\varphi')| \le \frac{1}{\varphi^2(1)}\, \|f'\|_\infty\, \|g'\|_\infty\, \|\varphi'\|_\infty \int_a^b w(x)\, H^2(x)\, dx,$$

where $H(x) = \int_a^b |P_{w,\varphi}(x,t)|\, dt$ and $\|\varphi'\|_\infty = ess \sup_{t\in[0,1]} |\varphi'(t)|$.

Since the functions f and g satisfy the hypothesis of Theorem 2.1, the following identities hold:

$$f(x) = \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt + \frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\, f'(t)dt \tag{2.8}$$

and

$$g(x) = \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt + \frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\, g'(t)\, dt\,. \tag{2.9}$$

Using (2.8) and (2.9) we obtain,

$$\begin{aligned}
&\left[f(x) - \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt\right] \\
&\qquad \times \left[g(x) - \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt\right] \\
&\qquad\qquad = \frac{1}{\varphi^2(1)}\left[\int_a^b P_{w,\varphi}(x,t)\, f'(t)dt\right]\left[\int_a^b P_{w,\varphi}(x,t)\, g'(t)\, dt\right].
\end{aligned}$$

Consequently,

$$f(x)\,g(x)-\frac{1}{\varphi(1)}f(x)\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)g(t)\,dt \tag{2.10}$$
$$-\frac{1}{\varphi(1)}g(x)\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt+\frac{1}{\varphi^2(1)}\left(\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt\right)$$
$$\times\left(\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)g(t)\,dt\right)$$
$$=\frac{1}{\varphi^2(1)}\left[\int_a^b P_{w,\varphi}(x,t)\,f'(t)dt\right]\left[\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt\right].$$

Multiplying both sides of (2.10) by $w(x)\,\varphi'\left(\int_a^x w(s)\,ds\right)$ and then integrating the resultant identity with respect to x from a to b, we get,

$$T(w,f,g,\varphi')=\frac{1}{\varphi^2(1)}\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) \tag{2.11}$$
$$\times\left[\int_a^b P_{w,\varphi}(x,t)\,f'(t)dt\right]\left[\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt\right]dx.$$

Finally,

$$|T(w,f,g,\varphi')|\le\frac{1}{\varphi^2(1)}\|f'\|_\infty\|g'\|_\infty\|\varphi'\|_\infty\int_a^b w(x)\,H^2(x)\,dx.$$

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