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ON A NEW STRENGTHENED VERSION OF A HARDY-HILBERT TYPE INEQUALITY AND APPLICATIONS

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Abstract

By improving an inequality of the weight coefficient, we give a new strengthened version of Hardy-Hilbert's type inequality. As its applications, we build some strengthened versions of the equivalent form and some particular results.

2000 Mathematics Subject Classification: 26D15.

Key words: Hardy-Hilbert's inequality, Weight coefficient, Hölder's inequality.

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1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then we have the famous Hardy-Hilbert inequality as follows [1]:

$$(1.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible.

Inequality (1.1) is important in analysis and its applications. In recent years, [2] – [5] considered the strengthened version, generalizations and improvements of inequality (1.1) and Pachpatte [6] built some inequalities similar to inequality (1.1).

Under the same condition with (1.1), we still have Mulholand’s inequality (cf. [7]):

$$(1.2) \quad \sum_{n=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=2}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible.

For the double series:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^n n^t (\ln m + \ln n + \ln \alpha)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^n n^t \ln \alpha m} \quad (s, t \in \mathbb{R}, \alpha \geq e^{7/6}),$$



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in 2003, Yang [8] built an inequality of the weight coefficient as follows:

$$\sum_{m=1}^{\infty} \frac{1}{m \ln \alpha mn} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha m}} \right)^{\frac{1}{r}} < \frac{\pi}{\sin \pi(1-1/r)} \quad (r > 1, \alpha \geq e^{7/6}),$$

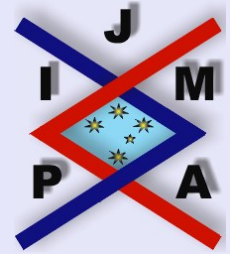
then he gave

$$(1.3) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^s n^t (\ln \alpha mn)} < \frac{\pi}{\sin \left(\frac{\pi}{p} \right)} \left(\sum_{n=1}^{\infty} \left(n^{\frac{1}{q}-s} a_n \right)^p \right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} \left(n^{\frac{1}{p}-t} b_n \right)^q \right)^{\frac{1}{p}},$$

$$(1.4) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin \left(\frac{\pi}{p} \right)} \right]^p \sum_{n=1}^{\infty} \left(n^{\frac{1}{q}-s} a_n \right)^p,$$

where the constant factors $\frac{\pi}{\sin \left(\frac{\pi}{p} \right)}$ and $\left[\frac{\pi}{\sin \left(\frac{\pi}{p} \right)} \right]^p$ are the best possible.

In this paper, by using the refined Euler-Maclaurin formula, we have some strengthened versions of inequalities (1.3) and (1.4).



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2. Some Lemmas

If $f^{(4)} \in C[1, \infty)$, $\int_1^\infty f(x)dx < \infty$, and $(-1)^n f^{(n)}(x) > 0$, $f^{(n)}(\infty) = 0$ ($n = 0, 1, 2, 3, 4$), then we have the following inequality (see [9]):

$$(2.1) \quad \sum_{m=1}^{\infty} f(m) < \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1).$$

Lemma 2.1. Setting $r > 1$, $n \in \mathbb{N}$ and $\alpha \geq e^{7/6}$, define the function $\omega(r, n, \alpha)$ as:

$$\omega(r, n, \alpha) = \sum_{m=1}^{\infty} \frac{1}{m \ln \alpha mn} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha m}} \right)^{\frac{1}{r}}.$$

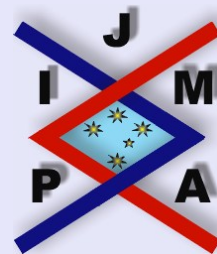
The we have

$$(2.2) \quad \omega(r, n, \alpha) < \frac{\pi}{\sin \pi(1 - 1/r)} - \frac{3}{8(r - 1)(2 \ln n + 1)^{1-1/r}}.$$

Proof. For fixed $x \in [1, \infty)$, setting $f(x) = \frac{1}{x \ln \alpha nx} \cdot \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha x}} \right)^{\frac{1}{r}}$, we have

$$f(1) = \frac{1}{\ln \alpha n} \cdot \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{\frac{1}{r}} = \frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha} \cdot \ln \alpha n} \cdot \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}},$$

$$f'(1) = - \left(\frac{1}{\ln \alpha n} + \frac{1}{\ln^2 \alpha n} + \frac{1}{r \ln \sqrt{\alpha} \cdot \ln \alpha n} \right) \cdot \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{1-\frac{1}{r}}$$



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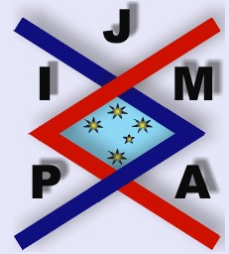
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$$= - \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha n} \cdot \ln \alpha n} + \frac{\ln \sqrt{\alpha n}}{r \ln^2 \sqrt{\alpha} \cdot \ln \alpha n} + \frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha} \cdot \ln^2 \alpha n} \right) \cdot \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}},$$

$$\begin{aligned} \int_1^\infty f(x) dx &= \int_1^\infty \frac{1}{x \ln \alpha n x} \cdot \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha x}} \right)^{\frac{1}{r}} dx \\ &= \int_{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}}^\infty \frac{1}{1+u} \cdot \left(\frac{1}{u} \right)^{\frac{1}{r}} du \\ &= \frac{\pi}{\sin(\pi/r)} - \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{1+u} \cdot \left(\frac{1}{u} \right)^{\frac{1}{r}} du. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{1+u} \cdot \left(\frac{1}{u} \right)^{\frac{1}{r}} du &= \frac{r}{r-1} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{1+u} \cdot du^{1-\frac{1}{r}} \\ &= \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \cdot \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r}{r-1} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} u^{1-\frac{1}{r}} \cdot \frac{1}{(1+u)^2} du \\ &= \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \cdot \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r^2}{(r-1)(2r-1)} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{(1+u)^2} du^{2-\frac{1}{r}} \\ &> \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \cdot \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r^2}{(r-1)(2r-1)} \end{aligned}$$



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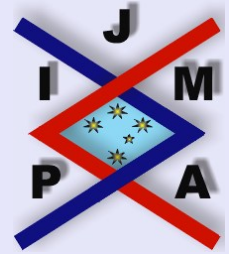
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$$\begin{aligned} & \cdot \left(\frac{\ln\sqrt{\alpha}}{\ln\alpha}n\right)^2 \cdot \left(\frac{\ln\sqrt{\alpha}}{\ln\alpha}n\right)^{2-\frac{1}{r}} \\ &= \left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha n}}\right)^{1-\frac{1}{r}} \left[\frac{r \ln\sqrt{\alpha n}}{(r-1)\ln\alpha n} + \frac{r^2}{(r-1)(2r-1)} \cdot \frac{\ln\sqrt{\alpha} \cdot \ln\sqrt{\alpha n}}{\ln^2\alpha n} \right], \end{aligned}$$

in view of (2.1) and the above result, we have

$$\begin{aligned} \omega(r, n, \alpha) &= \sum_{m=1}^{\infty} f(m) \\ &< \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1) \\ &< \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha n}}\right)^{1-\frac{1}{r}} \\ &\quad \cdot \left[\frac{r \ln\sqrt{\alpha n}}{(r-1)\ln\alpha n} + \frac{r^2}{(r-1)(2r-1)} \cdot \frac{\ln\sqrt{\alpha} \cdot \ln\sqrt{\alpha n}}{\ln^2\alpha n} \right] \\ &\quad + \frac{\ln\sqrt{\alpha n}}{2\ln\sqrt{\alpha} \cdot \ln\alpha n} \cdot \left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha n}}\right)^{1-\frac{1}{r}} + \frac{1}{12} \left(\frac{\ln\sqrt{\alpha n}}{\ln\sqrt{\alpha} \cdot \ln\alpha n}\right) + \frac{\ln\sqrt{\alpha n}}{r \ln^2\sqrt{\alpha} \cdot \ln\alpha n} \\ &\quad + \frac{\ln\sqrt{\alpha n}}{\ln\sqrt{\alpha} \cdot \ln^2\alpha n} \cdot \left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha n}}\right)^{1-\frac{1}{r}} \\ &= \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha n}}\right)^{1-\frac{1}{r}} \cdot \left\{ \left(\frac{r}{r-1} - \frac{7}{6\ln\alpha} - \frac{1}{3r\ln^2\alpha}\right) \cdot \frac{\ln\sqrt{\alpha n}}{\ln\alpha n} \right. \end{aligned}$$



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$$+ \left[\frac{r^2 \ln \alpha}{2(r-1)(2r-1)} - \frac{1}{6 \ln \alpha} \right] \cdot \frac{\ln \sqrt{\alpha n}}{\ln^2 \alpha n} \Bigg\}.$$

For $n \in \mathbb{N}$, $r > 1$, $\alpha \geq e^{7/6}$, since

$$\begin{aligned} \left(\frac{r}{1-r} - \frac{7}{6 \ln \alpha} - \frac{1}{3r \ln^2 \alpha} \right) \cdot \frac{\ln \sqrt{\alpha n}}{\ln \alpha n} &\geq \left(1 + \frac{1}{r-1} - \frac{7}{6 \cdot \frac{7}{6}} - \frac{1}{3r \cdot (\frac{7}{6})^2} \right) \cdot \frac{1}{2} \\ &= \frac{3}{8(r-1)}, \end{aligned}$$

$$\begin{aligned} &\left[\frac{r^2 \ln \alpha}{2(r-1)(2r-1)} - \frac{1}{6 \ln \alpha} \right] \cdot \frac{\ln \sqrt{\alpha n}}{\ln^2 \alpha n} \\ &> \left[\frac{\ln \alpha}{4} \left(1 + \frac{3}{2(r-1)} \right) - \frac{1}{6 \ln \alpha} \right] \cdot \frac{\ln \sqrt{\alpha n}}{\ln^2 \alpha n} \\ &> \left[\frac{7}{24} \left(1 + \frac{3}{2(r-1)} \right) - \frac{1}{7} \right] \cdot \frac{\ln \sqrt{\alpha n}}{\ln^2 \alpha n} > 0, \end{aligned}$$

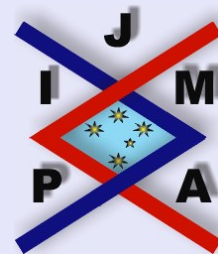
and

$$\left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} > \frac{1}{(2 \ln n + 1)^{1-1/r}},$$

we have

$$\omega(r, n, \alpha) < \frac{\pi}{\sin \pi(1-1/r)} - \frac{3}{8(r-1)(2 \ln n + 1)^{1-1/r}}.$$

The lemma is proved. □



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3. Main Results and Applications

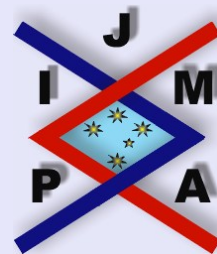
Theorem 3.1. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq e^{7/6}$, $s, t \in \mathbb{R}$, $a_n b_n$ are two sequences of non-negative real numbers, such that $0 < \sum_{n=1}^{\infty} \left(n^{\frac{1}{q}-s} a_n\right)^p < \infty$ and $0 < \sum_{n=1}^{\infty} \left(n^{\frac{1}{p}-t} b_n\right)^q < \infty$, then we have

$$(3.1) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^s n^t \ln \alpha mn} < \left[\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2 \ln n + 1)^{\frac{1}{p}}} \left(n^{\frac{1}{q}-s} a_n\right)^p \right)^{\frac{1}{p}} \right. \\ \left. \times \left[\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(q-1)}{8(2 \ln n + 1)^{\frac{1}{q}}} \right) \left(n^{\frac{1}{p}-t} b_n\right)^q \right]^{\frac{1}{q}} \right].$$

In particular,

(a) for $s = \frac{1}{q}, t = \frac{1}{p}$, we have

$$(3.2) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{\frac{1}{p}} n^{\frac{1}{q}} \ln \alpha mn} < \left[\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2 \ln n + 1)^{\frac{1}{p}}} \right) a_n^p \right]^{\frac{1}{p}}$$



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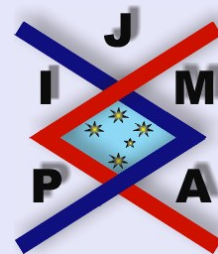
$$\times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n + 1)^{\frac{1}{q}}} \right) b_n^q \right]^{\frac{1}{q}} ;$$

(b) for $s = t = 1$, we have

$$(3.3) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m n \ln \alpha m n} < \left[\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n + 1)^{\frac{1}{p}}} \right) \frac{a_n^p}{n} \right]^{\frac{1}{p}} \times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n + 1)^{\frac{1}{q}}} \right) \frac{b_n^q}{n} \right]^{\frac{1}{q}}$$

(c) for $s = t = 0$, we have

$$(3.4) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln \alpha m n} < \left[\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n + 1)^{\frac{1}{p}}} \right) n^{p-1} a_n^p \right]^{\frac{1}{p}}$$



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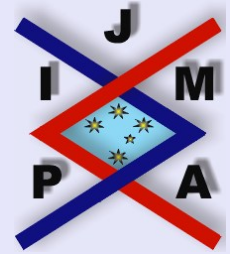
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$$\times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n + 1)^{\frac{1}{q}}} \right) n^{q-1} b_n^q \right]^{\frac{1}{q}}.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^s n^t \ln \alpha mn} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(\ln \alpha mn)^{\frac{1}{p}}} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{pq}} \left(\frac{m^{\frac{1}{q}-s}}{n^{\frac{1}{p}}} \right) \right] \\ & \quad \times \left[\frac{b_n}{(\ln \alpha mn)^{\frac{1}{q}}} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha m}} \right)^{\frac{1}{pq}} \left(\frac{n^{\frac{1}{p}-t}}{m^{\frac{1}{q}}} \right) \right] \\ &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{a_m^p}{\ln \alpha mn} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{q}} \left(\frac{m^{p(\frac{1}{q}-s)}}{n} \right) \right]^{\frac{1}{p}} \\ & \quad \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{b_n^q}{\ln \alpha mn} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha m}} \right)^{\frac{1}{p}} \left(\frac{n^{q(\frac{1}{p}-t)}}{m} \right) \right]^{\frac{1}{q}} \\ &= \left[\sum_{m=1}^{\infty} \omega(q, m, \alpha) \left(m^{\frac{1}{q}-s} a_m \right)^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \omega(p, n, \alpha) \left(n^{\frac{1}{p}-t} b_n \right)^q \right]^{\frac{1}{q}}. \end{aligned}$$

In view of (2.2), we have (3.1). The theorem is proved. \square



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Theorem 3.2. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq e^{7/6}$, $s \in \mathbb{R}$, a_n is sequence of non-negative real numbers, such that $0 < \sum_{n=1}^{\infty} \left(n^{\frac{1}{q}-s} a_n \right)^p < \infty$, then we have

$$(3.5) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2 \ln m + 1)^{\frac{1}{p}}} \right) \left(m^{\frac{1}{q}-s} a_m \right)^p.$$

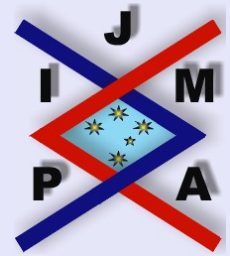
In particular,

(a) for $s = \frac{1}{q}$, we have

$$(3.6) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\frac{1}{q}} \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2 \ln m + 1)^{\frac{1}{p}}} \right) (a_m)^p;$$

(b) for $s = 1$, we have

$$(3.7) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2 \ln m + 1)^{\frac{1}{p}}} \right) \frac{a_m^p}{m};$$



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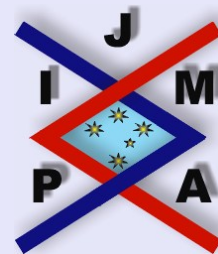
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(c) for $s = 0$, we have

$$(3.8) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2 \ln m + 1)^{\frac{1}{p}}} \right) (m^{p-1} a_m)^p.$$

Proof. It is obvious that for any $m \in N_0$, $\omega(r, m, \alpha) < \frac{\pi}{\sin \pi(1-1/r)}$. By Cauchy's inequality, we obtain

$$\begin{aligned} & \left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p \\ &= \left[\sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)^{\frac{1}{p}}} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{pq}} \left(m^{\frac{1}{q}-s} \right) a_m \right. \\ & \quad \left. \cdot \frac{1}{(\ln \alpha mn)^{\frac{1}{q}}} \left(\frac{\ln \sqrt{\alpha} n}{\ln \sqrt{\alpha} m} \right)^{\frac{1}{pq}} \left(\frac{1}{m^{\frac{1}{q}}} \right) \right]^p \\ &\leq \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \\ & \quad \cdot \sum_{m=1}^{\infty} \left[\frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} n}{\ln \sqrt{\alpha} m} \right)^{\frac{1}{p}} \frac{1}{m} \right]^{(p-1)} \end{aligned}$$



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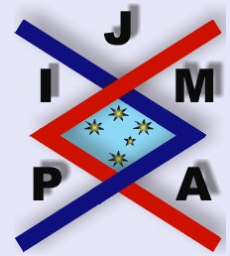
$$\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \cdot [\omega(p, n, \alpha)]^{(p-1)} \\
&< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p.
\end{aligned}$$

Hence, by (2.2) we find

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p \\
&< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \\
&= \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \ln \alpha mn} \left(\frac{\ln \sqrt{\alpha m}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \\
&= \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \omega(q, m, \alpha) \left(m^{\frac{1}{q}-s} a_m \right)^p \\
&< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin \frac{\pi}{p}} - \frac{3(p-1)}{8(2 \ln m + 1)^{\frac{1}{p}}} \right) \left(m^{\frac{1}{q}-s} a_m \right)^p.
\end{aligned}$$

The theorem is proved. \square

Remark 1. Obviously, inequalities (3.1) and (3.5) are separately strengthened versions of inequalities (1.3) and (1.4).



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- [1] F.H. HARDY AND J.E. LITTLEWOOD, *Inequalities*, London, Cambridge Univ. Press, 1952.
- [2] B.C. YANG AND L. DEBNATH, On a new generalization of Hardy-Hilbert's inequality and its applications., *J. Math. Anal. Appl.*, **233** (1999), 484–497.
- [3] B.C. YANG, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Mathematica Sinica*, **43**(6) (1999), 1003–1110 (in Chinese).
- [4] B.C. YANG, On a strengthened Hardy-Hilbert's inequality, *J. Ineq. Pure and Appl. Math.*, **1**(2) (2000), Art. 22. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=116>].
- [5] JICHANG KUANG AND L. DEBNATH, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, **245** (2000), 248–265.
- [6] B.G. PACHPATTE, On some inequalities similar to Hilbert's inequality, *J. Math. Anal.*, **226** (1998), 166–179.
- [7] H.P. MULHOLAND, A further generalization of Hilbert's double series theorem, *J. London Math. Soc.*, **6** (1931), 100–106.
- [8] B.C. YANG, A new Hardy-Hilbert's type inequality and Applications, *Acta Mathematica Sinica*, **46**(6) (2003), 1079–1086 (in Chinese).



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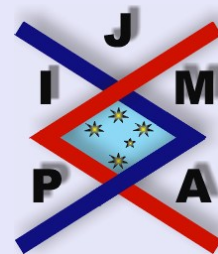
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- [9] JICHANG KUANG AND L. DEBNATH, On new generalization of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, **245** (2000), 248–265.



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