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HADAMARD PRODUCT OF CERTAIN MEROMORPHIC $p-$VALENT STARLIKE AND $p-$VALENT CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some results concerning the Hadamard product of certain meromorphic p-valent starlike and meromorphic p-valent convex functions analogous to those obtained by Vinod Kumar (J. Math. Anal. Appl. 113(1986), 230-234) and M. L. Mogra (Tamkang J. Math. 25(1994), no. 2, 157-162).

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1. INTRODUCTION

Throughout this paper, let $p \in \mathbb{N} = \{1, 2, \dots\}$ and let the functions of the form:

$$\varphi(z) = c_p z^p - \sum_{n=1}^{\infty} c_{p+n} z^{p+n} \qquad (c_p > 0; c_{p+n} \geq 0),$$

$$\Psi(z) = d_p z^p - \sum_{n=1}^{\infty} d_{p+n} z^{p+n} \qquad (d_p > 0; d_{p+n} \geq 0)$$

be regular and $p-$valent in the unit disc $U = \{z : |z| < 1\}$. Also, let

$$f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (a_{p-1} > 0; a_{p+n-1} \geq 0), \tag{1.1}$$

$$f_i(z) = \frac{a_{p-1,i}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,i} z^{p+n-1} \qquad (a_{p-1,i} > 0; a_{p+n-1,i} \geq 0),$$

$$g(z) = \frac{b_{p-1}}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1} z^{p+n-1} \qquad (b_{p-1} > 0; b_{p+n-1} \geq 0)$$

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and

$$g_j(z) = \frac{b_{p-1,j}}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1,j} z^{p+n-1} \qquad (b_{p-1,j} > 0; b_{p+n-1,j} \geq 0).$$

be regular and $p-$valent in the punctured disc $D = \{z : 0 < |z| < 1\}$.

Let $S_0^*(p, \alpha, \beta)$ denote the class of functions $\varphi(z)$ which satisfy the condition

$$\left| \frac{\frac{z\varphi'(z)}{\varphi(z)} - p}{\frac{z\varphi'(z)}{\varphi(z)} + p - 2\alpha} \right| < \beta$$

for some $\alpha, \beta\, (0 \leq \alpha < p,\, 0 < \beta \leq 1,\, p \in \mathbb{N})$ and for all $z \in U$; and let $C_0(p, \alpha, \beta)$ be the class of functions $\varphi(z)$ for which $\frac{zf'(z)}{p} \in S_0^*(p, \alpha, \beta)$. It is well known that the functions in $S_0^*(p, \alpha, \beta)$ and $C_0^*(p, \alpha, \beta)$ are, respectively, $p-$valent starlike and $p-$valent convex functions of order α and type β with negative coefficients in U (see Aouf [1]).

Denote by $\Sigma S_0^*(p, \alpha, \beta)$, the class of functions $f(z)$ which satisfy the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} + 2\alpha - p} \right| < \beta \tag{1.2}$$

for some $\alpha, \beta\, (0 \leq \alpha < p,\, 0 < \beta \leq 1,\, p \in \mathbb{N})$ and for all $z \in D$, and $\Sigma C_0^*(p, \alpha, \beta)$ be the class of functions $f(z)$ for which $\frac{-zf'(z)}{p} \in \Sigma S_0^*(p, \alpha, \beta)$. The functions in $\Sigma S_0^*(p, \alpha, \beta)$ and $\Sigma C_0^*(p, \alpha, \beta)$ are, respectively, called meromorphic $p-$valent starlike and meromorphic $p-$valent convex functions of order α and type β with positive coefficients in D. The class $\Sigma S_0^*(p, \alpha, \beta)$ with $a_{p-1} = 1$ has been studied by Aouf [2] and Mogra [9].

Using similar arguments as given in ([2] and [9]), we can prove the following result for functions in $\Sigma S_0^*(p, \alpha, \beta)$.

A function $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} \{[(n + 2p - 1) + \beta(n + 2\alpha - 1)]\, a_{n+p-1}\} \leq 2\beta(p - \alpha)a_{p-1}\,. \tag{1.3}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2\beta(p - \alpha)a_{p-1}}{(n + 2p - 1) + \beta(n + 2\alpha - 1)} z^{p+n-1} \quad (p, n \in \mathbb{N}).$$

Proof Outline. Let $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$ be given by (1.1). Then, from (1.2) and (1.1), we have

$$\left| \frac{\sum_{n=1}^{\infty} (n + 2p - 1) a_{p+n-1} z^{2p+n-1}}{2(p - \alpha)a_{p-1} - \sum_{n=1}^{\infty} (n + 2\alpha - 1) a_{p+n-1} z^{2p+n-1}} \right| < \beta \quad (z \in U). \tag{1.4}$$

Since $|\mathrm{Re}(z)| \leq |z|\ (z \in \mathbb{C})$, choosing z to be real and letting $z \to 1^-$ through real values, (1.4) yields

$$\sum_{n=1}^{\infty} (n + 2p - 1) a_{p+n-1} \leq 2\beta(p - \alpha)a_{p-1} - \sum_{n=1}^{\infty} \beta(n + 2\alpha - 1) a_{p+n-1}\,,$$

which leads us to (1.3).

In order to prove the converse, we assume that the inequality (1.3) holds true. Then, if we let $z \in \partial U$, we find from (1.1) and (1.3) that

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} + 2\alpha - p} \right| \leq \frac{\sum_{n=1}^{\infty} (n+2p-1) a_{p+n-1}}{(p-\alpha)a_{p-1} - \sum_{n=1}^{\infty}(n+2\alpha-1)a_{p+n-1}}$$
$$< \beta \quad (z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}).$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$. This completes the proof of (1.3). □

Also we can prove that $f(z) \in \Sigma C_0^*(p, \alpha, \beta)$ if and only if

$$\text{(1.5)} \qquad \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p} \right) [(n+2p-1) + \beta(n+2\alpha-1)]\, a_{n+p-1} \right\} \leq 2\beta(p-\alpha)a_{p-1}\,.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2\beta(p-\alpha)a_{p-1}}{\left(\frac{n+2p-1}{p}\right)[(n+2p-1) + \beta(n+2\alpha-1)]} z^{p+n-1} \quad (p, n \in \mathbb{N})\,.$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa ([11], [12] and [13]), Kumar ([6], [7] and [8]), Aouf et al. [3], Hossen [5], Darwish [4] and Sekine [14]. Accordingly, the quasi-Hadamard product of two functions $\varphi(z)$ and $\Psi(z)$ is defined by

$$\varphi * \Psi(z) = c_p d_p z^p - \sum_{n=1}^{\infty} c_{p+n} d_{p+n} z^{p+n}\,.$$

Let us define the Hadamard product of two functions $f(z)$ and $g(z)$ by

$$f * g(z) = \frac{a_{p-1}b_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} b_{p+n-1} z^{p+n-1}\,.$$

Similarly, we can define the Hadamard product of more than two meromorphic $p-$valent functions.

We now introduce the following class of meromorphic $p-$valent functions in D.

A function $f(z) \in \Sigma_k^*(p, \alpha, \beta)$ if and only if

$$\text{(1.6)} \qquad \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p} \right)^k [(n+2p-1) + \beta(n+2\alpha-1)]\, a_{n+p-1} \right\} \leq 2\beta(p-\alpha)a_{p-1}\,.$$

where $0 \leq \alpha < p$, $0 < \beta \leq 1$, $p \in \mathbb{N}$, and k is any fixed nonnegative real number.

Evidently, $\Sigma_0^*(p, \alpha, \beta) \equiv \Sigma S_0^*(p, \alpha, \beta)$ and $\Sigma_1^*(p, \alpha, \beta) \equiv \Sigma C_0^*(p, \alpha, \beta)$. Further, $\Sigma_k^*(p, \alpha, \beta) \subset \Sigma_h^*(p, \alpha, \beta)$ if $k > h \geq 0$, the containment being proper. Moreover, for any positive integer k, we have the following inclusion relation

$$\Sigma_k^*(p, \alpha, \beta) \subset \Sigma_{k-1}^*(p, \alpha, \beta) \subset \cdots \subset \Sigma_2^*(p, \alpha, \beta) \subset \Sigma C_0^*(p, \alpha, \beta) \subset \Sigma S_0^*(p, \alpha, \beta)\,.$$

We also note that for every nonnegative real number k, the class $\Sigma_k^*(p, \alpha, \beta)$ is nonempty as the functions

$$f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} \left(\frac{p+n-1}{p} \right)^{-k} \left\{ \frac{2\beta(p-\alpha)}{(n+2p-1) + \beta(n+2\alpha-1)} \right\} a_{p-1} \lambda_{p+n-1} z^{p+n-1},$$

where $a_{p-1} > 0$, $0 \leq \alpha < p$, $0 < \beta \leq 1$, $p \in \mathbb{N}$, $a_{p-1} > 0$, $\lambda_{p+n-1} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n-1} \leq 1$, satisfy the inequality (1.1).

In this paper we establish certain results concerning the Hadamard product of meromorphic $p-$valent starlike and meromorphic $p-$valent convex functions of order α and type β analogous to Kumar [7] and Mogra [10].

2. THE MAIN THEOREMS

Theorem 2.1. *Let the functions $f_i(z)$ belong to the class $\Sigma C_0^*(p,\alpha,\beta)$ for every $i=1,2,\dots,m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\Sigma_{2m-1}^*(p,\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m-1}[(n+2p-1)+\beta(n+2\alpha-1)]\prod_{i=1}^{m}a_{p+n-1,i}\right\} \leq 2\beta(p-\alpha)\left[\prod_{i=1}^{m}a_{p-1,i}\right].$$

Since $f_i(z)\in\Sigma C_0^*(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)[(n+2p-1)+\beta(n+2\alpha-1)]\,a_{p+n-1,i}\right\}\leq 2\beta(p-\alpha)a_{p-1,i}, \tag{2.1}$$

for $i=1,2,\dots,m$. Therefore,

$$\left(\frac{p+n-1}{p}\right)[(n+2p-1)+\beta(n+2\alpha-1)]\,a_{p+n-1,i}\leq 2\beta(p-\alpha)a_{p-1,i}$$

or

$$a_{p+n-1,i}\leq\left[\frac{2\beta(p-\alpha)}{\left(\frac{p+n-1}{p}\right)[(n+2p-1)+\beta(n+2\alpha-1)]}\right]a_{p-1,i}\,,$$

for every $i=1,2,\dots,m$. The right-hand expression of the last inequality is not greater than $\left(\frac{p+n-1}{p}\right)^{-2}a_{p-1,i}$. Hence

$$a_{p+n-1,i}\leq\left(\frac{p+n-1}{p}\right)^{-2}a_{p-1,i}, \tag{2.2}$$

for every $i=1,2,\dots,m$.

Using (2.2) for $i=1,2,\dots,m-1$, and (2.1) for $i=m$, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m-1}[(n+2p-1)+\beta(n+2\alpha-1)]\prod_{i=1}^{m}a_{p+n-1,i}\right\}\\&\leq\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m-1}[(n+2p-1)+\beta(n+2\alpha-1)]\right.\\&\qquad\left.\times\left(\left(\frac{p+n-1}{p}\right)^{-2(m-1)}\cdot\prod_{i=1}^{m-1}a_{p-1,i}\right)a_{p+n-1,m}\right\}\end{aligned}$$

$$= \left[\prod_{i=1}^{m-1} a_{p-1,i}\right] \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right) [(n+2p-1)+\beta(n+2\alpha-1)]\, a_{n+p-1,m} \right\}$$
$$\leq 2\beta(p-\alpha) \left[\prod_{i=1}^{m} a_{p-1,i}\right] .$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \Sigma^*_{2m-1}(p,\alpha,\beta)$. □

Theorem 2.2. *Let the functions $f_i(z)$ belong to the class $\Sigma S_0^*(p,\alpha,\beta)$ for every $i = 1,2,\ldots,m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\Sigma^*_{m-1}(p,\alpha,\beta)$.*

Proof. Since $f_i(z) \in \Sigma S_0^*(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty} \{[(n+2p-1)+\beta(n+2\alpha-1)]\, a_{p+n-1,i}\} \leq 2\beta(p-\alpha) a_{p-1,i}, \tag{2.3}$$

for $i = 1,2,\ldots,m$. Therefore,

$$a_{p+n-1,i} \leq \left\{ \frac{2\beta(p-\alpha)}{[(n+2p-1)+\beta(n+2\alpha-1)]} \right\} a_{p-1,i}\, ,$$

and hence

$$a_{p+n-1,i} \leq \left(\frac{p+n-1}{p}\right)^{-1} a_{p-1,i}, \tag{2.4}$$

for every $i = 1,2,\ldots,m$.

Using (2.4) for $i = 1,2,\ldots,m-1$, and (2.3) for $i = m$, we get

$$\sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right)^{m-1} [(n+2p-1)+\beta(n+2\alpha-1)] \prod_{i=1}^{m} a_{p+n-1,i} \right\}$$
$$\leq \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right)^{m-1} [(n+2p-1)+\beta(n+2\alpha-1)] \right.$$
$$\left. \times \left(\left(\frac{p+n-1}{p}\right)^{-(m-1)} \cdot \prod_{i=1}^{m-1} a_{p-1,i} \right) a_{p+n-1,m} \right\}$$
$$= \left[\prod_{i=1}^{m-1} a_{p-1,i}\right] \sum_{n=1}^{\infty} \{[(n+2p-1)+\beta(n+2\alpha-1)]\, a_{n+p-1,m}\}$$
$$\leq 2\beta(p-\alpha) \left[\prod_{i=1}^{m} a_{p-1,i}\right].$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \Sigma^*_{m-1}(p,\alpha,\beta)$. □

Theorem 2.3. *Let the functions $f_i(z)$ belong to the class $\Sigma C_0^*(p,\alpha,\beta)$ for every $i = 1,2,\ldots,m$, and let the functions $g_j(z)$ belong to the class $\Sigma S_0^*(p,\alpha,\beta)$ for every $j = 1,2,\ldots,q$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\Sigma^*_{2m+q-1}(p,\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m+q-1}[(n+2p-1)+\beta(n+2\alpha-1)] \times\left(\prod_{i=1}^{m} a_{p+n-1,i}\cdot\prod_{i=1}^{q} b_{p+n-1,i}\right)\right\} \leq 2\beta(p-\alpha)\left(\prod_{i=1}^{m} a_{p-1,i}\prod_{i=1}^{q} b_{p-1,i}\right).$$

Since $f_i(z) \in \Sigma C_0^*(p,\alpha,\beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1, 2, \ldots, m$. Further, since $g_j(z) \in \Sigma S_0^*(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\left\{[(n+2p-1)+\beta(n+2\alpha-1)]\, b_{p+n-1,j}\right\} \leq 2\beta(p-\alpha) b_{p-1,j}, \tag{2.5}$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$b_{p+n-1,j} \leq \left(\frac{p+n-1}{p}\right)^{-1} b_{p-1,j}\ , \tag{2.6}$$

for every $j = 1, 2, \ldots, q$.

Using (2.2) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q-1$, and (2.5) for $j = q$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m+q-1}[(n+2p-1)+\beta(n+2\alpha-1)] \times\left(\prod_{i=1}^{m} a_{p+n-1,i}\cdot\prod_{j=1}^{q} b_{p+n-1,j}\right)\right\}\\
&\leq \sum_{n=1}^{\infty}\left\{\left(\frac{p+n-1}{p}\right)^{2m+q-1}[(n+2p-1)+\beta(n+2\alpha-1)]\right.\\
&\qquad\left.\times\left(\left(\frac{p+n-1}{p}\right)^{-2m}\left(\frac{p+n-1}{p}\right)^{-(q-1)}\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q-1} b_{p-1,j}\right) b_{p+n-1,q}\right\}\\
&= \left(\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q-1} b_{p-1,j}\right)\sum_{n=1}^{\infty}\left\{[(n+2p-1)+\ \beta(n+2\alpha-1)]\, b_{p+n-1,q}\right\}\\
&\leq 2\beta(p-\alpha)\left(\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q} b_{p-1,j}\right).
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z) \in \Sigma^*_{2m+q-1}(p,\alpha,\beta)$. □

We note that the required estimate can also be obtained by using (2.2) for $i = 1, 2, \ldots, m-1$, (2.6) for $j = 1, 2, \ldots, q$, and (2.1) for $i = m$.

Remark 1. Putting $p = 1$ in the above results, we obtain the results obtained by Mogra [10].

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