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SOME P.D.F.-FREE UPPER BOUNDS FOR THE DISPERSION $\sigma(X)$ AND THE QUANTITY $\sigma^2(X) + (x - EX)^2$

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Abstract

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Abstract

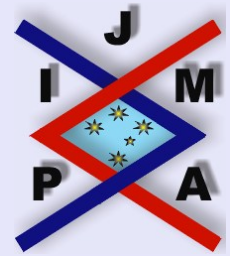
In comparison with Theorems 2.1 and 2.4 in [1], we provide some *p.d.f.*-free upper bounds for the dispersion $\sigma(X)$ and the quantity $\sigma^2(X) + (x - EX)^2$ taking only into account the endpoints of the given finite interval.

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1. Introduction and Results

Let $f : [a, b] \subset \mathbb{R} \rightarrow [0, \infty)$ be the *p.d.f.* (probability density function) of a random variable X whose expectation and dispersion are respectively given by

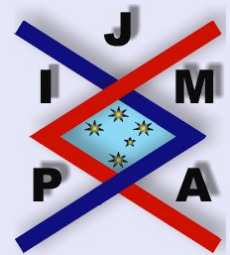
$$EX = \int_a^b t f(t) dt$$

and

$$\sigma(X) = \sqrt{\int_a^b (t - EX)^2 f(t) dt} = \sqrt{\int_a^b t^2 f(t) dt - (EX)^2}.$$

In [1], Theorems 2.1 and 2.4, the following upper bounds were obtained for the dispersion $\sigma(X)$

$$\sigma(X) \leq \begin{cases} \frac{\sqrt{3}(b-a)^2}{6} \|f\|_\infty & \text{if } f \in L_\infty[a, b] \\ \frac{\sqrt{2}(b-a)^{1+q^{-1}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}} \|f\|_p & \text{if } f \in L_p[a, b], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{\sqrt{2}(b-a)}{2} & \text{if } f \in L_1[a, b] \end{cases}$$



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and the quantity $\sigma^2(X) + (x - EX)^2$

$$\sigma^2(X) + (x - EX)^2 \leq \begin{cases} (b-a) \left[\frac{(b-a)^2}{12} + \left(x - \frac{b+a}{2}\right)^2 \right] \sqrt{\|f\|_\infty} & \text{if } f \in L_\infty[a, b] \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{2q}} \sqrt{\|f\|_p} & \text{if } f \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \left(\frac{b-a}{2} + \left|x - \frac{b+a}{2}\right|\right)^2 & \text{if } f \in L_1[a, b] \end{cases}$$

for all $x \in [a, b]$.

In this communication we intend to make free from the *p.d.f.* the above upper bounds for the dispersion $\sigma(X)$ and the quantity $\sigma^2(X) + (x - EX)^2$ taking only into account the endpoints of the given finite interval.

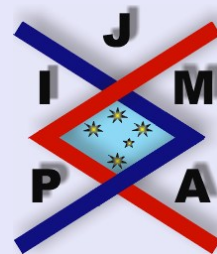
Theorem 1.1. *Under the above restriction on the p.d.f. we have*

$$\sigma(X) \leq \min \{ \max \{ |a|, |b| \}, b - a \}.$$

Proof. First, for any number $t \in [a, b]$ we note (via $f(t) \geq 0$) that $af(t) \leq tf(t) \leq bf(t)$ leading to $a \leq EX \leq b$. Consequently,

$$(1.1) \quad 0 \leq EX - a \leq b - a \quad \text{and} \quad 0 \leq b - EX \leq b - a.$$

We point out that the function $g : [a, b] \rightarrow [0, \infty)$, defined by $g(t) = (t - EX)^2$, is a bounded convex function which assumes the minimum at point $(EX, 0)$.



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Thus

$$\begin{aligned}\beta &:= \sup \{ (t - EX)^2 : t \in [a, b] \} \\ &= \max \{ (a - EX)^2, (b - EX)^2 \} \\ &\leq (b - a)^2,\end{aligned}$$

by taking into consideration (1.1). Now, it can be easily seen that

$$\sigma(X) = \sqrt{\int_a^b (t - EX)^2 f(t) dt} \leq \sqrt{\beta \int_a^b f(t) dt} = \sqrt{\beta} \leq b - a.$$

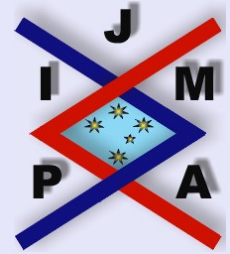
Next, using the facts that function $h(t) = t^2$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$ on the one hand and,

$$\sigma(X) = \sqrt{\int_a^b t^2 f(t) dt - (EX)^2} \leq \sqrt{\int_a^b t^2 f(t) dt}$$

on the other, we can easily check that

$$\sigma^2(X) \leq \int_a^b t^2 f(t) dt \leq \begin{cases} b^2 & \text{if } a \geq 0 \\ \max \{a^2, b^2\} & \text{if } a < 0 \text{ and } b > 0 \\ a^2 & \text{if } b \leq 0, \end{cases}$$

so that $\sigma^2(X) \leq \max \{a^2, b^2\}$. Therefore, we can conclude on the validity of the argument. \square



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Theorem 1.2. Under the above restriction on the p.d.f. we have

$$\sqrt{\sigma^2(X) + (x - EX)^2} \leq 2 \min \{ \max \{ |a|, |b| \}, b - a \}$$

for all $x \in [a, b]$.

Proof. We recall the identity

$$\sigma^2(X) + (x - EX)^2 = \int_a^b (t - x)^2 f(t) dt, \quad x \in [a, b],$$

from the proof of Theorem 2.4 in [1]. Clearly,

$$\int_a^b (t - x)^2 f(t) dt \leq \max \{ (t - x)^2 : t, x \in [a, b] \},$$

so that

$$\sqrt{\sigma^2(X) + (x - EX)^2} \leq \max \{ |t - x| : t, x \in [a, b] \}.$$

It is obvious that $0 \leq t - a \leq b - a$ and $0 \leq x - a \leq b - a$, since $t, x \in [a, b]$.

We note that we can estimate from above the quantity $|t - x|$ in two ways:

$$|t - x| \leq |t - a| + |a - x| \leq 2(b - a)$$

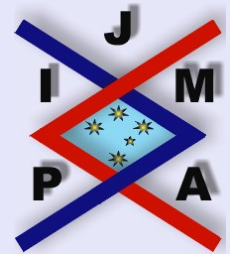
and

$$|t - x| \leq |t| + |x| \leq 2 \max \{ |a|, |b| \}.$$

Consequently,

$$\max \{ |t - x| : t, x \in [a, b] \} \leq 2 \min \{ \max \{ |a|, |b| \}, b - a \}.$$

This leads to the desired result. □



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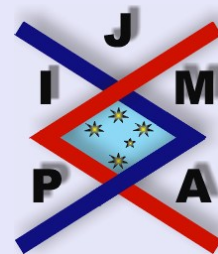
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