λ-CENTRAL BMO ESTIMATES FOR COMMUTATORS OF N-DIMENSIONAL HARDY OPERATORS

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Abstract: This paper gives the λ-central BMO estimates for commutators of n-dimensional Hardy operators on central Morrey spaces.

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1. Introduction and Main Results

Let f be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operators are defined by

$$\mathcal{H}f(x) := \frac{1}{|x|^n}\int_{|t|\le|x|} f(t)dt, \quad \mathcal{H}^*f(x) := \int_{|t|>|x|}\frac{f(t)}{|t|^n}dt, \qquad x\in\mathbb{R}^n\setminus\{0\}.$$

In [4], Christ and Grafakos obtained results for the boundedness of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ spaces. They also found the exact operator norms of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ spaces, where $1<p<\infty$.

It is easy to see that $\mathcal{H}$ and $\mathcal{H}^*$ satisfy

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}f(x)\,dx = \int_{\mathbb{R}^n} f(x)\mathcal{H}^*g(x)\,dx. \tag{1.1}$$

We have

$$|\mathcal{H}f(x)| \le C_n Mf(x),$$

where M is the Hardy-Littlewood maximal operator which is defined by

$$Mf(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(t)|dt, \tag{1.2}$$

where the supremum is taken over all balls containing x.

Recently, Fu et al. [2] gave the definition of commutators of n-dimensional Hardy operators.

Definition 1.1. *Let b be a locally integrable function on $\mathbb{R}^n$. We define the commutators of n-dimensional Hardy operators as follows:*

$$\mathcal{H}_b f := b\mathcal{H}f - \mathcal{H}(fb), \qquad \mathcal{H}_b^* f := b\mathcal{H}^* f - \mathcal{H}^*(fb).$$

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In [2], Fu et al. gave the central BMO estimates for commutators of n-dimensional Hardy operators. In 2000, Alvarez, Guzmán-Partida and Lakey [1] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces, respectively.

Definition 1.2 (λ-central BMO space). *Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < \frac{1}{n}$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ if*

$$\|f\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)} = \sup_{\mathrm{R}>0} \left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}} \int_{B(0,\mathrm{R})} |f(x) - f_{B(0,\mathrm{R})}|^q dx \right)^{\frac{1}{q}} < \infty. \tag{1.3}$$

Remark 1. *If two functions which differ by a constant are regarded as a function in the space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, then $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ becomes a Banach space. Apparently, (1.3) is equivalent to the following condition (see [1]):*

$$\sup_{\mathrm{R}>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}} \int_{B(0,\mathrm{R})} |f(x) - c|^q dx \right)^{\frac{1}{q}} < \infty.$$

Definition 1.3 (Central Morrey spaces, see [1]). *Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < 0$. The central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \sup_{\mathrm{R}>0} \left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}} \int_{B(0,\mathrm{R})} |f(x)|^q dx \right)^{\frac{1}{q}} < \infty. \tag{1.4}$$

Remark 2. *It follows from (1.3) and (1.4) that $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is a Banach space continuously included in $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$.*

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Inspired by [2], [3] and [5], we will establish the λ-central BMO estimates for commutators of n-dimensional Hardy operators on central Morrey spaces.

Theorem 1.4. *Let $\mathcal{H}_b$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0, 0 \le \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.4. We can obtain the central BMO estimates for commutators of n-dimensional Hardy operators, $\mathcal{H}_b$, on central Morrey spaces.

Corollary 1.5. *Let $\mathcal{H}_b$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ and $-\frac{1}{q} < \lambda < 0$. If $b \in C\dot{M}O^{p_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda}(\mathbb{R}^n)}.$$

Similar to Theorem 1.4, we have:

Theorem 1.6. *Let $\mathcal{H}_b^*$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0$, $0 \le \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b^*$ is bounded from $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.6. We can get the central BMO estimates for commutators of n-dimensional Hardy operators, $\mathcal{H}_b^*$, on central Morrey spaces.

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Corollary 1.7. *Let* $\mathcal{H}_b^*$ *be defined as above. Suppose* $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $-\frac{1}{q} < \lambda < 0$. *If* $b \in C\dot{M}O^{p_2}(\mathbb{R}^n)$, *then the commutator* $\mathcal{H}_b^*$ *is bounded from* $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *and satisfies the following inequality:*

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda}(\mathbb{R}^n)}.$$

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2. Proofs of Theorems

Proof of Theorem 1.4. Let f be a function in $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)$. For fixed $R>0$, denote $B(0,R)$ by B. Write

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |\mathcal{H}_b f(x)|^q dx\right)^{\frac{1}{q}}\\
&=\left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(x)-b(y))\,dy\right|^q dx\right)^{\frac{1}{q}}\\
&\leq \left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(x)-b_B)\,dy\right|^q dx\right)^{\frac{1}{q}}\\
&\qquad+\left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx\right)^{\frac{1}{q}}\\
&:=I+J.
\end{aligned}$$

For $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}$, by Hölder's inequality and the boundedness of $\mathcal{H}$ from L^{p_1} to L^{p_1}, we have

$$\begin{aligned}
I &\leq |B|^{-\frac{1}{q}}\left(\int_B |b(x)-b_B|^{p_2}dx\right)^{\frac{1}{p_2}}\left(\int_B |\mathcal{H}(f\chi_B)(x)|^{p_1}dx\right)^{\frac{1}{p_1}}\\
&\leq C|B|^{-\frac{1}{q}}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|B|^{\frac{1}{p_2}+\lambda_2}\left(\int_B |f(x)|^{p_1}dx\right)^{\frac{1}{p_1}}\\
&= C|B|^{\lambda}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\left(\frac{1}{|B|^{1+p_1\lambda_1}}\int_B |f(x)|^{p_1}dx\right)^{\frac{1}{p_1}}\\
&\leq C|B|^{\lambda}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.
\end{aligned}$$

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For J, we have

$$J^q = \frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx$$
$$= \frac{1}{|B|}\sum_{k=-\infty}^{0}\int_{2^kB\backslash 2^{k-1}B}\left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx$$
$$\leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b(y)-b_B)\,dy\right|^q dx$$
$$\leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b(y)-b_{2^iB})\,dy\right|^q dx$$
$$+ \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b_{2^iB}-b_B)\,dy\right|^q dx$$
$$:= J_1 + J_2$$

By Hölder's inequality ($\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q}$), we have

$$J_1 \leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{|2^kB|}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|2^iB|^{\frac{1}{q'}}\left(\int_{2^iB}|f(y)|^{p_1}\,dy\right)^{\frac{1}{p_1}}\right.$$
$$\left.\times\left(\int_{2^iB}|b(y)-b_{2^iB}|^{p_2}\,dy\right)^{\frac{1}{p_2}}\right\}^q$$
$$\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB|}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|2^iB|^{\lambda+1}\right\}^q$$

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$$\leq C|B|^{q\lambda}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

To estimate J_2, the following fact is applied.

For $\lambda_2 \geq 0$,

$$\begin{aligned}
|b_{2^iB} - b_B| &\leq \sum_{j=i}^{-1} |b_{2^{j+1}B} - b_{2^jB}| \\
&\leq \sum_{j=i}^{-1} \frac{1}{|2^jB|}\int_{2^jB} |b(y) - b_{2^{j+1}B}|dy \\
&\leq C\sum_{j=i}^{-1}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^{p_2}dy\right)^{\frac{1}{p_2}} \\
&\leq C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|B|^{\lambda_2}\sum_{j=i}^{-1}2^{(j+1)n\lambda_2} \\
&\leq C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|i||B|^{\lambda_2}.
\end{aligned}$$

By Hölder's inequality ($\frac{1}{p_1} + \frac{1}{p_1'} = 1$), we have

$$\begin{aligned}
J_2 &= \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\setminus 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\setminus 2^{i-1}B} f(y)(b_{2^iB}-b_B)\,dy\right|^q dx \\
&\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB||B|^{q\lambda_2}}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|i||2^iB|^{\lambda_1+1}\right\}^q \\
&\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB||B|^{q\lambda_2}|k|^q|2^kB|^{(\lambda_1+1)q}}{|2^kB|^q}
\end{aligned}$$

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$$\leq C|B|^{q\lambda}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Combining the estimates of I, J_1 and J_2, we get the required estimate for Theorem 1.4. □

Proof of Theorem 1.6. We omit the details here. □

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References

[1] J. ALVARAREZ, M. GUZMAN-PARTIDA AND J. LAKEY, Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures, *Collect. Math.*, **51** (2000), 1–47.

[2] Z.W. FU, Z.G. LIU, S.Z. LU AND H.B. WANG, Characterization for commutators of N-dimensional fractional Hardy operators. *Science in China (Ser. A)*, **10** (2007), 1418–1426.

[3] Z.W. FU, Y. LIN AND S.Z. LU, λ-Central BMO estimates for commutators of singular integral operators with rough kernels. *Acta Math. Sinica (English Ser.)*, **3** (2008), 373–386.

[4] M. CHRIST AND L. GRAFAKOS, Best constants for two non-convolution inequalities, *Proc. Amer. Math. Soc.*, **123** (1995), 1687–1693.

[5] S.C. LONG AND J. WANG, Commutators of Hardy operators, *J. Math. Anal. Appl.*, **274** (2002), 626–644.

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