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ON MULTIPLICATIVELY e-PERFECT NUMBERS

JÓZSEF SÁNDOR

DEPARTMENT OF MATHEMATICS
BABEŞ-BOLYAI UNIVERSITY
STR. KOGALNICEANU, 400084 CLUJ-NAPOCA
ROMANIA.
jsandor@math.ubbcluj.ro

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ABSTRACT. Let $T_e(n)$ denote the product of exponential divisors of n. An integer n is called multiplicatively e-perfect, if $T_e(n) = n^2$. A characterization of multiplicatively e-perfect and similar numbers is given.

Key words and phrases: Perfect number, exponential divisor, multiplicatively perfect, sum of divisors, number of divisors.

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1. INTRODUCTION

If $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is the prime factorization of $n > 1$, a divisor $d|n$, called an exponential divisor (e-divisor, for short), of n is $d = p_1^{b_1} \dots p_r^{b_r}$ with $b_i|\alpha_i$ $(i = \overline{1, r})$. This notion is due to E. G. Straus and M. V. Subbarao [11]. Let $\sigma_e(n)$ be the sum of divisors of n. For various arithmetic functions and convolutions on e-divisors, see J. Sándor and A. Bege [10]. Straus and Subbarao define n as exponentially perfect (or e-perfect for short) if

$$\sigma_e(n) = 2n. \tag{1.1}$$

Some examples of e-perfect numbers are: $2^2 \cdot 3^2$, $2^2 \cdot 3^3 \cdot 5^2$, $2^4 \cdot 3^2 \cdot 11^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2$, etc. If m is squarefree, then $\sigma_e(m) = m$, so if n is e-perfect, and $m =$ squarefree with $(m, n) = 1$, then $m \cdot n$ is e-perfect, too. Thus it suffices to consider only powerful (i.e. no prime occurs to the first power) e-perfect numbers.

Straus and Subbarao [11] proved that there are no odd e-perfect numbers, and that for each r the number of e-perfect numbers with r prime factors is finite.

Is there an e-perfect number which is not divisible by 3? Straus and Subbarao conjecture that there is only a finite number of e-perfect numbers not divisible by any given prime p.

J. Fabrykowski and M.V. Subbarao [3] proved that any e-perfect number not divisible by 3 must be divisible by 2^{117}, greater than 10^{664}, and have at least 118 distinct prime factors.

P. Hagis, Jr. [4] showed that the density of e-perfect numbers is positive.

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For results on e-multiperfect numbers, i.e. satisfying

$$\sigma_e(n) = kn \tag{1.2}$$

$(k > 2)$, see W. Aiello, G. E. Hardy and M. V. Subbarao [1]. See also J. Hanumanthachari, V. V. Subrahmanya Sastri and V. Srinivasan [5], who considered also e-superperfect numbers, i.e. numbers n satisfying

$$\sigma_e(\sigma_e(n)) = 2n. \tag{1.3}$$

2. MAIN RESULTS

Let $T(n)$ denote the *product* of divisors of n. Then n is said to be multiplicatively perfect (or m-perfect) if

$$T(n) = n^2 \tag{2.1}$$

and multiplicatively super-perfect, if

$$T(T(n)) = n^2.$$

For properties of these numbers, with generalizations, see J. Sándor [8].

A divisor d of n is said to be "unitary" if $\left(d, \frac{n}{d}\right) = 1$. Let $T^*(n)$ be the product of unitary divisors of n. A. Bege [2] has studied the multiplicatively unitary perfect numbers, and proved certain results similar to those of Sándor. He considered also the case of "bi-unitary" divisors.

The aim of this paper is to study the multiplicatively e-perfect numbers. Let $T_e(n)$ denote the product of e-divisors of n. Then n is called multiplicatively e-perfect if

$$T_e(n) = n^2, \tag{2.2}$$

and multiplicatively e-superperfect if

$$T_e(T_e(n)) = n^2. \tag{2.3}$$

The main result is contained in the following:

Theorem 2.1. *n is multiplicatively e-perfect if and only if $n = p^\alpha$, where p is a prime and α is an ordinary perfect number. n is multiplicatively e-superperfect if and only if $n = p^\alpha$, where p is a prime, and α is an ordinary superperfect number, i.e. $\sigma(\sigma(\alpha)) = 2\alpha$.*

Proof. First remark that if p prime,

$$T_e(p^\alpha) = \prod_{d|\alpha} p^\alpha = p^{\sum_{d|\alpha} d} = p^{\sigma(\alpha)}.$$

Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then the exponential divisors of n have the form $p_1^{d_1} \cdots p_r^{d_r}$ where $d_1|\alpha_1, \ldots, d_r|\alpha_r$. If $d_1, \ldots, d_{r-1}$ are fixed, then these divisors are $p_1^{d_1} \cdots p_{r-1}^{d_{r-1}} p_r^d$ with $d|\alpha_r$ and the product of these divisors is $p_1^{d_1 d(\alpha_r)} \cdots p_{r-1}^{d_{r-1} d(\alpha_r)} p_r^{\sigma(\alpha_r)}$, where $d(a)$ is the number of divisors of a, and $\sigma(a)$ denotes the sum of divisors of a. For example, when $r = 2$, we get $p_1^{d_1 d(\alpha_2)} p_2^{\sigma(\alpha_2)}$. The product of these divisors is $p_1^{\sigma(d_1) d(\alpha_2)} p_2^{\sigma(\alpha_2) d(\alpha_1)}$. In the general case (by first fixing $d_1, \ldots, d_{r-2}$, etc.), it easily follows by induction that the following formula holds true:

$$T_e(n) = p_1^{\sigma(\alpha_1) d(\alpha_2) \cdots d(\alpha_r)} \cdots p_r^{\sigma(\alpha_r) d(\alpha_1) \cdots d(\alpha_{r-1})} \tag{2.4}$$

Now, if n is multiplicatively e-perfect, by (2.2), and the unique factorization theorem it follows that

$$\begin{cases} \sigma(\alpha_1) d(\alpha_2) \cdots d(\alpha_r) = 2\alpha_1 \\ \cdots \\ \sigma(\alpha_r) d(\alpha_1) \cdots d(\alpha_{r-1}) = 2\alpha_r \end{cases}. \tag{2.5}$$

This is impossible if all $\alpha_i = 1$ $(i = \overline{1,r})$. If at least an $\alpha_i = 1$, let $\alpha_1 = 1$. Then $d(\alpha_2)\cdots d(\alpha_r) = 2$, so one of $\alpha_2, \ldots, \alpha_r$ is a prime, the others are equal to 1. Let $\alpha_2 = p$, $\alpha_3 = \cdots = \alpha_r = 1$. But then the equation $\sigma(\alpha_2)d(\alpha_1)d(\alpha_3)\cdots d(\alpha_r) = 2\alpha_2$ of (2.5) gives $\sigma(\alpha_2) = 2\alpha_2$, i.e. $\sigma(p) = 2p$, which is impossible since $p + 1 = 2p$.

Therefore, we must have $\alpha_i \geq 2$ for all $i = \overline{1,r}$.

Let $r \geq 2$ in (2.5). Then the first equation of (2.5) implies

$$\sigma(\alpha_1)d(\alpha_2)\cdots d(\alpha_r) \geq (\alpha_1 + 1)\cdot 2^{r-1} \geq 2(\alpha_1 + 1) > 2\alpha_1,$$

which is a contradiction. Thus we must have $r = 1$, when $n = p_1^{\alpha_1}$ and $T_e(n) = p_1^{\sigma(\alpha_1)} = n^{2\alpha_1}$ iff $\sigma(\alpha_1) = 2\alpha_1$, i.e. if α_1 is an ordinary perfect number. This proves the first part of the theorem.

By (2.4) we can write the following complicated formula:

$$(2.6)\quad T_e(T_e(n)) = p_1^{\sigma(\sigma(\alpha_1)d(\alpha_2)\cdots d(\alpha_r))\cdots d(\sigma(\alpha_r)d(\alpha_1)\cdots d(\alpha_{r-1}))} \cdots p_r^{\sigma(\sigma(\alpha_r)d(\alpha_1)\cdots d(\alpha_{r-1}))\cdots d(\sigma(\alpha_1)d(\alpha_2)\cdots d(\alpha_r))}.$$

Thus, if n is multiplicatively e-superperfect, then

$$(2.7)\quad \begin{cases} \sigma(\sigma(\alpha_1)d(\alpha_2)\cdots d(\alpha_r))\cdots d(\sigma(\alpha_r)d(\alpha_1)\cdots d(\alpha_{r-1})) = 2\alpha_1 \\ \cdots \\ \sigma(\sigma(\alpha_r)d(\alpha_1)\cdots d(\alpha_{r-1}))\cdots d(\alpha(\alpha_1)d(\alpha_2)\cdots d(\alpha_r)) = 2\alpha_r \end{cases}.$$

As above, we must have $\alpha_i \geq 2$ for all $i = 1, 2, \ldots, r$.

But then, since $\sigma(ab) \geq a\sigma(b)$ and $\sigma(b) \geq b+1$ for $b \geq 2$, (2.7) gives a contradiction, if $r \geq 2$. For $r = 1$, on the other hand, when $n = p_1^{\alpha_1}$ and $T_e(n) = p_1^{\sigma(\alpha_1)}$ we get $T_e(T_e(n)) = p_1^{\sigma(\sigma(\alpha_1))}$, and (2.3) implies $\sigma(\sigma(\alpha_1)) = 2\alpha_1$, i.e. α_1 is an ordinary superperfect number. □

Remark 2.2. No odd ordinary perfect or superperfect number is known. The even ordinary perfect numbers are given by the well-known Euclid-Euler theorem: $n = 2^k p$, where $p = 2^{k+1} - 1$ is a prime ("Mersenne prime"). The even superperfect numbers have the general form (given by Suryanarayana-Kanold [12], [6]) $n = 2^k$, where $2^{k+1} - 1$ is a prime. For new proofs of these results, see e.g. [7], [9].

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